

Course in Pisa

1-8 June 2012

SHAPE OPTIMIZATION

Speakers:

Michel Perine
Dorin Bocu
Bernard Kuvshinov

Organizers

Antonie Henrot
Giuseppe Buttazzo

Office hours $\rightarrow$ 17h40 - 19h20 (except Wednesday, Friday)

Colegio Pascano

}	Benoit : 14
	Antonia : 15
	Michel : 25
	Dorin : 26

AVOUR DORIN Existence of Optimal Shapes

* questions, elementary problems $\rightarrow$ 1st consequences



$f \in L^2(\mathbb{R}^2)$

$$\left\{ \begin{array}{l} -\Delta u_\epsilon = f \text{ in } \Omega_\epsilon \\ u_\epsilon = 0 \text{ on } \partial\Omega_\epsilon \end{array} \right.$$

- * relaxation
- * recent results on free discontinuity problems

KAWHL BERND Geometric properties of Optimal Shapes

- * field of rearrangement of function / convexity
- * minimal resistance, aerodynamic resistance
- * Cheeger sets / convex bodies of constant width
- www.mmi.uni-koeln.de / K Kawhl

PIERRE MICHEL Regularity of Optimal Shapes

- * 1st step in studying regularity
Optimal shapes comes with power regularity
regularity of minimal surfaces
geometric measure theory

- ① Minimizing perimeter $P(\Omega)$ with prescribed volume
- ② Shape optimization $G(\Omega) = G(\Omega, u_\Omega)$
 $\uparrow$ solution of a PDE
- ③ $G(\Omega) + P(\Omega)$
- ④ Shape optimization with convexity constraints

Lundi 04/06/12

DORIN BUCUR

11h - 12h30

Existence of optimal shapes

References: A. Henrot / M. Pierre → Variations et optimisation de forme
2005

G. Buttazzo / D. Bucur → Variational methods in shape optimization problems, 2005

Introduction: Geometric formulation $\min_{\Omega \in \mathcal{U}_{adm}} F(\Omega)$

- $\Omega \rightarrow$ open set
- $\rightarrow$ closed set
- $\rightarrow$ measurable set

$\mathcal{U}_{\text{adm}}$ = set of admissible shapes

→ all open sets

→ all convex sets

→ etc.

F = cost functional

$F = F(u, \Omega)$

↑ solution of the Dirichlet-Laplace problem

$F = F(h_1(\Omega), \dots, h_k(\Omega))$

Examples: ① The isoperimetric inequality

$\min_{|\Omega|=c} P(\Omega)$

solution $\Rightarrow$ ball (unique)

Ω open, smooth
(integral of a surface)

or Ω Lebesgue measurable
(generalized perimeter)

② $\Omega \subset \mathbb{R}^N$, Ω open measurable, $|\Omega| = c$

$\begin{cases} -\Delta u = \lambda_\Omega u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

resolvent is compact (due to the compactness of the embedding)

$H_0^1(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega)$

$\exists \quad 0 < \lambda_1(\Omega) < \dots < \lambda_k(\Omega) \xrightarrow[k \rightarrow \infty]{} +\infty$

$$\lambda_k(\Omega) = \min_{\substack{S_k \in H_0^1(\Omega) \\ \text{subspace of dimension } k}} \max_{u \in S_k} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}$$

$$\lambda_1(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}$$

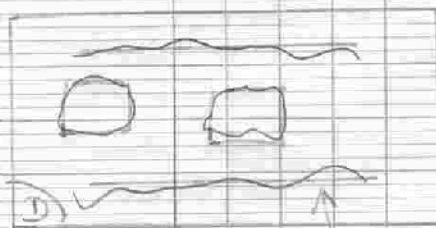
Rayleigh conjecture 1877 $\rightarrow \min_{|\Omega|=C} \lambda_1(\Omega) = \text{ball}$

solution given later by Faber-Krahn $\rightarrow$ simplest case of shape problem with PDE

③ The Mumford-Shah functional

(history of the problem related to De Giorgi in Pisa)

$f: D \rightarrow \mathbb{R}$ $f \in L^\infty(D)$
 D rectangle



We want to replace the level of gray by lines and then identify the region of constant level.

$$\min_{\Gamma} \min_{u \in H^1(D \setminus \Gamma)} \left(\int_{D \setminus \Gamma} |\nabla u|^2 + \int_D (u - f)^2 + \ell(\Gamma) \right)$$

$\int_{D \setminus \Gamma} |\nabla u|^2$: new region close to the original one
 $\int_D (u - f)^2$:
 $\ell(\Gamma)$: length of Γ

If Γ is fixed, then u solves $\begin{cases} -\Delta u + u = f & \text{in } D \setminus \Gamma \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \cup \Gamma \end{cases}$

Questions:

* Existence of a solution

* Qualitative properties

→ smoothness
→ symmetries
→ convex
→ ball

* Numerical computations

Remark: Sometimes, the solution does not exist?

Find best shape of ice to cold water as fast as possible?
You want to crash ice.

$$\max_{\Omega \subset \mathbb{R}^2} P(\Omega)$$

no $\exists$

How to prove existence? In general

* Take a minimizing sequence $(\Omega_n)_{n \in \mathbb{N}}$ ($\exists$ by definition of inf)

* Try to construct the best shape as a limit of a subsequence (Ω_{n_k})

- By compactness $\Omega_{n_k} \rightarrow \Omega^*$ ($\Omega^* \in \mathcal{H}_m$?)

* Lower semi-continuity $F(\Omega^*) \leq \liminf_{k \rightarrow \infty} F(\Omega_{n_k})$
(I.S.C.)

We need a topology in the family of shape
 diffeomorphisms $\rightarrow$ find the right topology

The requirements lower semi-continuity + compactness are opposite
 $\uparrow$ $\uparrow$
 need many open sets need many compact

Examples:

$$\min_{\substack{|u| \leq c \\ r \subseteq D \\ \uparrow \\ \text{open bounded box}}} h(r) = \min_{\substack{r \subseteq D \\ |u| \leq c \\ r \subseteq D}} \min_{u \in H^1(r)} \frac{\int_D |\nabla u|^2}{\int_D |u|^2}$$

Here, it is easy to rephrase the problem in terms of function
 (or in any other)

$$\min_{\substack{|u| \leq c \\ r \subseteq D}} h(r) = \min_{\substack{u \in H^1(D) \\ |\{u > 0\}| \leq c}} \frac{\int_D |\nabla u|^2}{\int_D |u|^2} \quad \text{and } D = \{u > 0\}$$

Take a minimizing sequence (u_n) $\left\{ \begin{array}{l} u_n \geq 0 \\ \int_D u_n^2 = 1 \\ \|u_n\|_{H^1(D)} \leq M \\ |\{u_n > 0\}| \leq c \\ D_n = \{u_n > 0\} \end{array} \right.$

We get $\lim_{n \rightarrow \infty} \frac{H_0(D)}{n} = u$

$$\int_D u^2 = 1$$

$$\int_D |Du|^2 \leq \liminf \int_D |Du_n|^2$$

(l.s.c. of the norm)

$$\{u > 0\} \subseteq C$$

$$\frac{\int_D |Du|^2}{\int_D u^2} \leq \liminf \frac{\int_D |Du_n|^2}{\int_D u_n^2} \Rightarrow \Omega = \{u > 0\}$$

Ω open? No, in general

Which representative? The best continuous

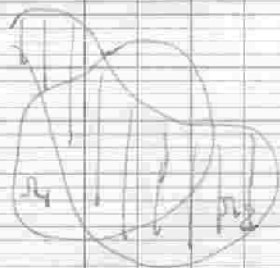
Only quasi-open (local set of Sobolev functions)

Then, regularity, it is open and smooth in fact

$$\lim_{n \rightarrow \infty} \frac{H_0(D)}{n} = u$$

Tentatives of topology:

* L^1 -topology of characteristic functions (topology of measurable sets)



$$\begin{aligned} d_{\text{char}}(\Omega_1, \Omega_2) &= \int_{\mathbb{R}^n} |1_{\Omega_1} - 1_{\Omega_2}| dx \\ &= \lambda(\Omega_1 \Delta \Omega_2) \end{aligned}$$

compactness: $\{\Omega = D, \text{Per}(\Omega, D) \leq C\}$ good for isoperimetric inequalities

→ does not work for PDEs

B		$\begin{cases} -\Delta u = 1 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$	solutions are $\neq$ but the sets are the same (Lebesgue measure) → most capacity
B \setminus \Gamma		$\begin{cases} -\Delta u = 1 & \text{in } B \setminus \Gamma \\ u = 0 & \text{on } \partial B \cup \Gamma \end{cases}$	

* Hausdorff distance $K_1, K_2 \subseteq \mathbb{R}^N$ closed sets

$$d_H(K_1, K_2) = \|d_{K_1}(\cdot) - d_{K_2}(\cdot)\|_{L^\infty}$$

Ω_1, Ω_2 open

$$d_H(\Omega_1, \Omega_2) = d_H(\Omega_1^c, \Omega_2^c)$$

compactness: $\{ \Omega \subseteq D, \Omega \text{ open} \}$ is compact for d_H
 (consequence of Arzelà-Ascoli theorem)

→ Pfm: when does the solution of the PDE continuous in d_H ?

Sverak 1992 (d-1)-rect Yes, if finite connected components of the complementary

Calibrated examples:

① Caramazza-Murat 1979 → "the strange term coming from somewhere else?"

(3)



$$\Omega_m = e^{-cm^2}$$

$C > 0$ fixed constant given
 $f \in L^2(D)$ fixed

$$\Omega_m = \Omega \setminus \bigcup_{i,j=1}^m \overline{B}_{1/n}(x_{ij})$$

We have: $|\Omega \setminus \Omega_m| \leq \text{area } m^2 e^{-cm^2} \xrightarrow{m \rightarrow \infty} 0$

$$\begin{cases} -\Delta u_m = f & \text{in } \Omega_m \\ u_m \in H_0^1(\Omega_m) \end{cases} \quad \exists M > 0, \|u_m\|_{H_0^1(\Omega)} \leq M$$

so $\{u_m\}_{m \in \mathbb{N}} \subset H_0^1(\Omega)$ for $m \geq N$
 because $u_m \in H_0^1(\Omega_m)$ for $m \geq N$
 because $\Omega_m \subset \Omega$.

But we find that u solves $\begin{cases} -\Delta u + \underbrace{Cu}_{\text{strange term}} = f & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$

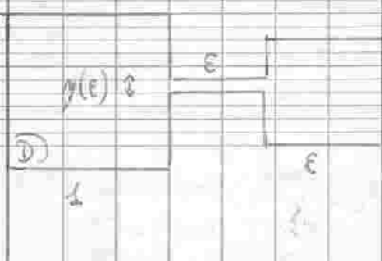
Exercise: Introduce the periodic function $\begin{cases} z_m \in H^1(D) \\ \Delta z_m = 0 \text{ on } B(x_{ij}, \frac{1}{n}) \setminus B(x_{ij}, \frac{1}{2n}) \\ z_m = 0 \end{cases}$



Take $\phi \in H_0^1(\Omega)$ and use ϕz_m as test function in Ω_m .

Remark: This phenomenon is related to relaxation problem (Dal'Arco...)

② The Courant-Hilbert problem:



$$g(E) \sim E^3$$

$$\begin{cases} -\Delta u = g u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \end{cases}$$

$$\mu_0(\Omega_\epsilon) = \mu_0(\Omega) = 0 \quad (\Omega = \mathbb{D})$$

$$\mu_1(\Omega) = \pi^2$$

$$\mu_1(\Omega_\epsilon) \rightarrow \text{any } c \in [0, \pi^2] \text{ playing with } \eta(\epsilon)$$

Other examples $\rightarrow$ Babuška paradox $\rightarrow$ $\Delta^{\frac{n-2}{2}}$ order equation on disk (problem in finite elements)
approximated by polygons

Regularity effects in fluid dynamics $\rightarrow$ Navier Stokes
 $\rightarrow$ sharp skin

Microscopic asperities $\rightarrow$ no-slip condition changes at the limit

II/ Geometric domain perturbation for the Dirichlet-Laplacian:

D bounded open box

$f \in L^2(D)$ fixed

$\Omega_\epsilon \subseteq D$ sequence of open sets

We consider the problem

$$\begin{cases} -\Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

$\Updownarrow$ Lax-Milgram theorem

Here $H_0^1(\Omega_\epsilon) = \overline{C_0^\infty(\Omega_\epsilon)}$

This is not true for quasi-open sets

$u_\epsilon \in H_0^1(\Omega_\epsilon)$ such that

$$\forall \Phi \in H_0^1(\Omega_\epsilon), \int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \Phi = \int_{\Omega_\epsilon} f \Phi$$

$\Updownarrow$

$u_\epsilon \in H_0^1(\Omega_\epsilon)$ minimizes in $H_0^1(\Omega_\epsilon)$ the functional
 $\frac{1}{2} \int_{\Omega_\epsilon} |\nabla \phi|^2 - \int_{\Omega_\epsilon} f \phi$

Remarks: * Let (Ω_ϵ) is increasing in the sense of inclusion in $\mathbb{D}$
 Let us define:

$$\Omega = \bigcup_{\epsilon} \Omega_{\epsilon}$$

Thm: $u_{\epsilon} \xrightarrow{H_0^1(\mathbb{D})} u_n$ (solution on Ω)

Proof: Indeed, we have $\|u_{\epsilon}\|_{H_0^1(\mathbb{D})} \leq C \Rightarrow u_{\epsilon_n} \xrightarrow{H_0^1(\mathbb{D})} u$

The purpose is $u = u_n$? If $u \in H_0^1(\Omega)$, yes because $\Omega_{\epsilon} \subseteq \Omega$ and choice of $H_0^1(\Omega)$

Take $\varphi \in C_c^\infty(\Omega)$ For sufficiently great ϵ , we have:
 $\text{supp } \varphi \subseteq \Omega_{\epsilon}$

Then, we get: $\int_{\Omega_{\epsilon}} \nabla u_{\epsilon} \nabla \varphi = \int_{\Omega_{\epsilon}} f \varphi$

$$\downarrow$$

$$\int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f \varphi \quad \text{uniqueness gives } u = u_n$$

Moreover the convergence is strong $\Rightarrow$ use u_{ϵ} as test function to get
 $\int_{\Omega_{\epsilon}} |\nabla u_{\epsilon}|^2 = \int_{\Omega_{\epsilon}} f u_{\epsilon} \Rightarrow \int_{\Omega_{\epsilon}} |\nabla u_{\epsilon}|^2 = \int_{\Omega} f u_{\epsilon}$

The full sequence (u_{ϵ}) converges because bounded + limit unique.

* Now, let (Ω_{ϵ}) be a decreasing sequence. Define: $\Omega = \bigcap_{\epsilon} \Omega_{\epsilon}$

(u_{ϵ}) bounded in $H_0^1(\mathbb{D}) \Rightarrow (u_{\epsilon_n}) \xrightarrow{H_0^1(\mathbb{D})} u$

Pass to the limit in the equation $\Rightarrow \forall \varphi \in C_c^\infty(\Omega)$, $\int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f \varphi$

Fundamental question $\Rightarrow$ Does $u \in H_0^1(\Omega)$?

* Take $\Omega_\varepsilon = \Omega + B_\varepsilon$. We have strong Δ almost everywhere

$$u_\varepsilon \rightarrow u \Rightarrow u = 0 \text{ a.e. in ext } \Omega$$

* If Ω is convex $\rightarrow$ yes also



$$u = 0 \text{ on ext } \Omega$$

Here it is true that $u \in H^1_0(\Omega)$



* For general Ω , the answer is NO! (Notion of stable set)

Lundi 06/06/12

BERND WANSCH

15h30 - 17h00

Geometric properties of Optimal Shapes

The hotspot conjecture of J. Rauch



$$u - \Delta u = 0 \quad (\Omega \in \mathbb{R}^n) \times \mathbb{R}^+$$

Oftenly after some time we get

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+$$

$$u(x, 0) = u_0(x)$$

$$\{x \mid u(x, t) = \max_{y \in \Omega} u(y, t)\}$$

$$\in \partial \Omega$$



1-D case

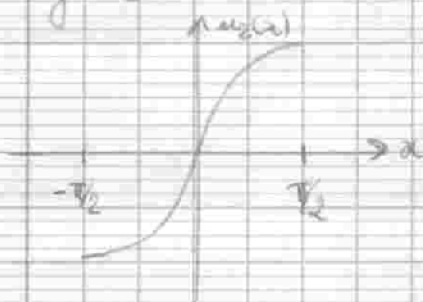
$$\text{Take } u(x, t) = \sum_{j=1}^{+\infty} a_j u_j(x) e^{-\lambda_j t} \quad \text{where } \begin{cases} -\Delta u_j = \lambda_j u_j \text{ on } \Omega \\ \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \partial \Omega \end{cases}$$

$$D_1 = 0 \quad u_1 = \text{cte}$$

$$\text{If } a_1 = 0 \quad \text{NO}$$



If $a_2 \neq 0$ and V_2 greater so preserve shape longer.



Conjecture: Show that u_2 attains its maximum on $\partial\Omega$.

u_2 can be characterized by minimizing the Rayleigh quotient

$$R(u) = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} |u|^2} \quad \text{on } W^{1,2}(\Omega) \cap \left\{ \int_{\Omega} u = 0 \right\}$$

↑ orthogonal to a constant

Hope is: if u_2 has local maximum in Ω rearrange it so that $R(u_2) > R(u_2^*)$
into Ω^* but $\int_{\Omega^*} u_2^* = 0 \Rightarrow u_2$

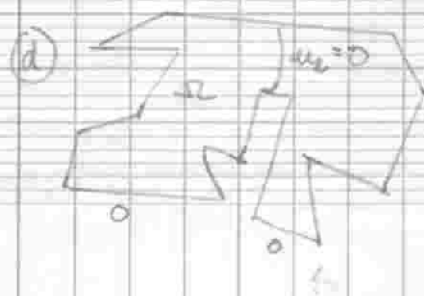


The conjecture is true and proved

(a) if Ω is cylindrical (→ cylinder in particular) (1)

(b) if Ω is long and thin → 1989 Bammann/Barday (1)
PARABOLIC PROOF

(c) if Ω is convex and has 2 axes of reflection symmetry → also Jensen + Nirenberg



car = Renault

Numerical tests

(c) There are also counterexamples $\rightarrow$ Schoen/Berestycki
for multiply connected domain

Now, consider the wave equation with Dirichlet boundary conditions

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times \mathbb{R}_+ \\ u = 0 & \partial\Omega \times \mathbb{R}_+ \\ u = u_0 \text{ initially} \end{cases} \quad u(x,t) = \sum_{j=1}^{+\infty} a_j \varphi_j(x) e^{-\lambda_j t}$$

The shape of $u(x,t)$ when $t \rightarrow +\infty$ is determined by the shape of u_0 provided the initial data:

$$\int_{\Omega} u_0(x) \varphi_1(x) > 0$$

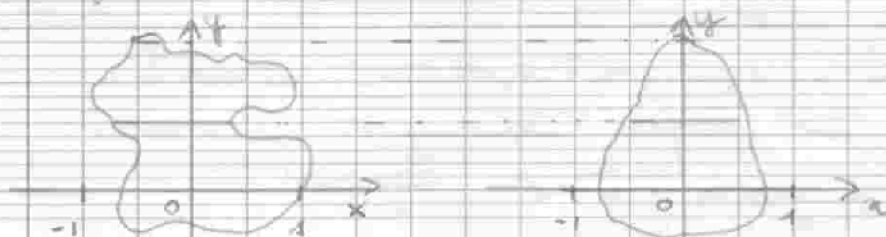
Then if  at $t=0$, then after  when $t \rightarrow +\infty$

How is the shape of the first Dirichlet eigenfunction determined by the shape of Ω ?

(a) If Ω is a ball, then u_1 is radial and radially decreasing (level function of radius)

(b) If Ω has some reflection symmetry in x and convex in direction x , then so are the level sets $\Omega_s = \{x \in \Omega \mid u(x) \geq s\}$

Definition: Steiner symmetric in x



(Thm: If u is optimal and exists, then it is Steiner symmetric.)

Idea of proof: If u^* represents the Steiner symmetrisation of u and u is Steiner symmetric, $u \geq 0$

$$\text{Then } \int_{\Omega} u^2 = \int_{\Omega^*} u^{*2} \text{ and } \int_{\Omega^*} |\nabla u^*|^2 \leq \int_{\Omega} |\nabla u|^2$$

equality holds if and only if $u = u^*$

So, if u_1 is the first Dirichlet eigenfunction

$$\frac{\int_{\Omega} |\nabla u_1|^2}{\int_{\Omega} u_1^2} \leq \lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla u_1|^2}{\int_{\Omega} u_1^2} = \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2} \leq \frac{\int_{\Omega} |\nabla u^*|^2}{\int_{\Omega} u^{*2}}$$

Idea for proving @:

(a) Moving plane method



Nice paper

Gruber, Nirenberg

(a) If u is not radial then there exists a direction, x says, such that u is not Steiner symmetric with respect to x .
(Contradiction by @ Like solution should be Steiner symmetric)

(a) Use the fact that u_1 is unique (modulo scaling). If u_1 is not radial, rotate it to get some $\tilde{u}_1$ such that $u_1 \neq \tilde{u}_1$.

1st Proof of uniqueness: If two different solutions exist, their difference must change sign. By linear combination, the difference is also an eigenfunction and so does not change sign. Contradiction.

If not, $\alpha u_1 + \beta \tilde{u}_1$ has to change sign, which contradicts the positivity of the 1st eigenfunction.

2nd proof for uniqueness: $\lambda(u) = \inf_{v \in V_0^p(u)} \frac{\ln |u|_p}{\ln |v|_p}$

first regeneration of the p. *Lophium*

This is a non-linear proof $\rightarrow$ substitute $v = u^p$

If u and $\bar{u}$ are eigenfunction, take $w = \left(\frac{1}{2}u^p + \frac{1}{2}\bar{u}^p \right)^{1/p}$

$$\|u\|_2 = 1 \quad \text{if} \quad \|u\|_2 = \|\tilde{u}\|_2 = 1$$

$$\int |\nabla u|^p = \int |\nabla u|^p h(u) \quad \text{so } R(u) = \int |\nabla u|^p h(u) \\ \text{strictly convex in } u \quad \leq \int (R(u) + R(v)) \\ \leq 2(u, v)$$

→ Lindquist + Krawohl

© If $\mathcal{R}$ is convex, so are the level sets of u .

Idea of proof: $u(\frac{1}{2}x_1 + \frac{1}{2}x_2) \leq \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2)$
 $\uparrow$ convexity
 $\forall (x_1, x_2) \in \mathbb{R}^2$

u_1 is not concave but $\log u_1$ is concave!

Instead, $\sigma = \log u$, $\tau = \frac{Du}{u}$, $\Delta \sigma = \frac{1}{u^2} (u \Delta u - Du^2)$

11. points $\Delta G = -\lambda \cdot |\vec{v}_0|^2 \ll 0$

Why is it concave? $C(a, x) = \underbrace{f(x)}_{\text{concave}} - \underbrace{f(a)}_{\text{const}} - \underbrace{f'(a)(x-a)}_{\text{linear}}$
 ≤ 0 in $\Omega \times \mathbb{R}$?

If u has finite maximum in $(x_1, x_2) \in \Omega \cap D$ (maximum principle argument)

$$\begin{cases} D_x u = 0 \\ D_y u = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2} u(x_1) = \frac{1}{2} u(x_2) \\ \frac{1}{2} u(x_1) = \frac{1}{2} u(x_2) \end{cases}$$

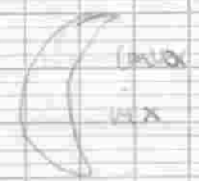
or $\tilde{u} = u(x_1 + y_1, x_2 + y_2) = \tilde{u}(y_1, y_2)$ has maximum at $(0,0)$

2nd derivative $\Delta u(x_1) \leq \frac{1}{2} \Delta u(x_1) + \frac{1}{2} \Delta u(x_2)$

$$\dots \leq 0 \quad \text{Kronecker}$$

④ If Ω is convex in x , then the level sets are not convex in x

can $\rightarrow$ J. Webb



Definition: If u has the property that its level sets are convex, then u is called quasi-concave



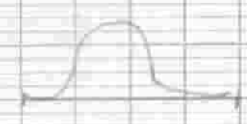
$$u\left(\frac{x_1 + x_2}{2}\right) \geq \min(u(x_1), u(x_2))$$

Respectively quasi-convexity / quasi-concavity of solution to PDE is to show:

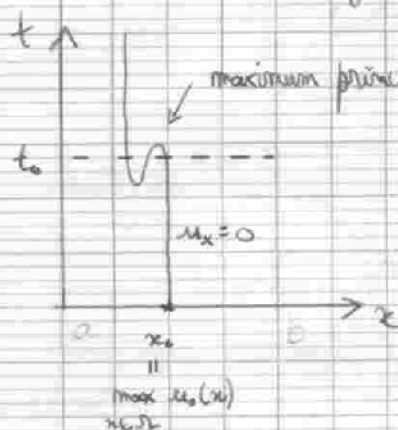
$$D(u, u) = u(x_1) - \min(u(x_1), u(x_2)) \geq 0 \quad \text{on } \Omega \cap D$$

Gabriel-Kohn method

in 1-D quasi-concave is called unimodal



Heat equation: if u_0 is quasi-concave, is $u(x,t)$ for each time t quasi-concave?



The answer is YES

UNIMODAL

"QUASI-CONCAVE 1D"

If u remains not unimodal

$$u_x = 0$$

$u \equiv \text{cte}$ in a shaded window

because

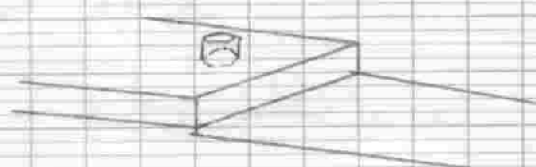
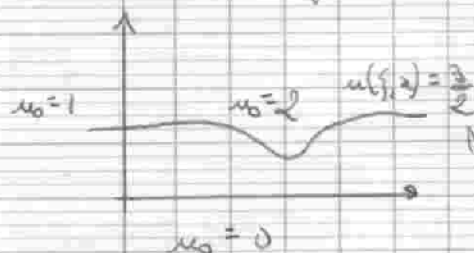
$w(t) = \#$ of zeros of u_x at time t is decreasing in time

"Hopf number result of Nagata (but Sturm)"

$$\begin{cases} u_t = \Delta u & \Omega \times \mathbb{R}_+ \\ u = 0 & \partial\Omega \times \mathbb{R}_+ \\ u = u_0 & \text{initially} \end{cases}$$

It is not true in general that $\overline{u}(\cdot, t)$ is quasi-concave

Counterexample $\rightarrow$ 2008 Shige + Salari



this level line is not convex

The isobut problem: Ω convex

$$\min_{\substack{u \geq \psi \\ u \in H_0^1(\Omega)}} \int_{\Omega} |\nabla u|^2$$



Conjecture: u is quasi-concave

2 partial results: (a) If ψ is χ_C (characteristic function of a convex) then Gabriel-Lewis result

(b) If ψ is quadratic polynomial with $\psi = 0$ on $\partial\Omega$, then $\chi \in \mathcal{E}$.

what is known? u has level sets that are starshaped with respect to the maximum point of ψ .



NOT STARSHAPED

PROOF: Show that $(x - x_0) \cdot \nabla u < 0$
 u is radially decreasing from x_0 .

We know that $(x - x_0) \cdot \nabla \psi < 0$ so $v(x) := (x - x_0) \cdot \nabla u \leq 0$ on $\partial\Omega$ because $u \geq 0$ in Ω and Ω is starshaped with respect to x_0 .

$(x - x_0) \cdot \nabla u = (x - x_0) \cdot \nabla \psi$ on coincidence set, then it must be ≤ 0 .

$u \in C^1(\Omega) \Rightarrow (x - x_0) \cdot \nabla u \leq 0$ on $\partial\Omega \cup \text{coincidence set}$

Then $\Delta v = 2\Delta u + (a - a_0)\mathbb{D}(\Delta u) = 0$ in $(u > \psi)$

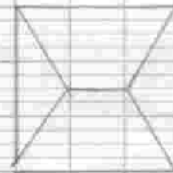
v harmonic function $\Rightarrow v \leq 0$ in $(u > \psi)$

* In general $\rightarrow$ if Ω has a symmetry
 u solution of a variational problem

then does u have a symmetry?

Examples of a variational problem with Ω reflection symmetric
 but the solution has only one reflectⁿ symmetry.

Opaque square problem:



Steiner tree
 geometry of 4 points

$|\Omega^*| \approx 2,8$

2,7...



shortest doubling
 connected fence

2,6...

Mardi 05/06/12 / REGULARITY FOR ISOPERIMETRIC PROBLEMS (without proofs):

MICHEL PIERRE

9:00-10:30

$D \subseteq \mathbb{R}^N$ open bounded box
1.1 Lebesgue measure in $\mathbb{R}^N$

Definitions:

(P) ① $\min_{\substack{\Omega \subseteq D \\ \text{measurable}}} P(\Omega) = P(\hat{\Omega})$

$|\Omega| = m$ where $0 < m \leq |D|$

we have $\exists$ if bounded,
what about the regularity
of $\hat{\Omega}$?

generalized perimeter of Di Giorgi $P(\Omega) = \sup \left\{ \int_{\Omega} \operatorname{div} \varphi, \varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N), \|\varphi\|_{L^\infty} \leq 1 \right\}$

$$\sup_{\varphi \in \mathbb{R}^N} \sqrt{\sum_{i=1}^N \varphi_i^2(x)}$$

Remark: If Ω is regular, then $\int_{\Omega} \operatorname{div} \varphi = \int_{\partial \Omega} \varphi \cdot \nu \, d\sigma$

We have then $1 \leq \|\varphi\|_{L^\infty} \int_{\partial \Omega} d\sigma \leq \int_{\partial \Omega} d\sigma$

it is easy to choose φ as an extension of ν on $\mathbb{R}^N$.

↑ in fact one splits

relative perimeter * If U is open $\subseteq \mathbb{R}^N$, then $P(U, \Omega) = \sup \left\{ \int_{\Omega} \operatorname{div} \varphi, \varphi \in C_c^\infty(U, \mathbb{R}^N), \|\varphi\|_{L^\infty} \leq 1 \right\}$
If Ω is regular then $P(\Omega, U) = \int_{\partial \Omega \cap U} d\sigma$

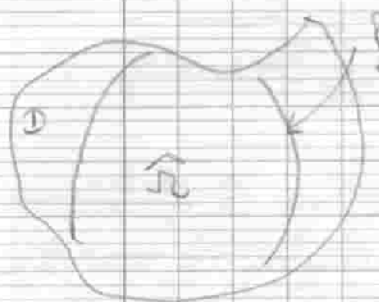


② $\min_{\substack{\Omega \subseteq D \\ \text{measurable} \\ |\Omega| = m}} P(\Omega, D)$

For ①, if D is large enough, then $\bar{\Omega} \equiv \text{ball}$ because of the isoperimetric inequality

$$P(\Omega) \geq C_N |\Omega|^{\frac{N-1}{N}}$$

↑
equality holds for ball



Theorem: Let $\hat{\Omega}$ be an optimal solution for ① or ②, then

* if $N \leq 7$, $\partial \hat{\Omega} \cap D$ is an analytic hypersurface

* $\hat{\Omega} \cap D$ is an analytic hypersurface
reduced boundary

$$\forall \delta > N-8, \mathcal{H}^S((\hat{\Omega} \setminus \hat{\Omega}^\delta) \cap D) = 0.$$

Some facts from measure theory: L.C. Evans R.F. Gariepy
 Theory of fine properties of functions
 O.K. Breda

* $0 \leq \lambda \leq N, \delta > 0 \quad E \subseteq \mathbb{R}^N \quad \alpha(\lambda) = \pi^{\frac{\lambda}{2}} \Gamma(\frac{\lambda}{2}) / \Gamma(\frac{\lambda}{2} + 1)$

$$H_\delta^\lambda(E) = \inf \left\{ \alpha(\lambda) \sum_i (\text{diam } E_i)^\lambda : E \subseteq \bigcup_i E_i, \text{diam } E_i \leq \delta \right\}$$

↑
 make sure that $\lambda = N$ Lebesgue measure
 $\lambda = N-1$ surface area

$$\mathcal{H}^\lambda(E) = \lim_{\delta \rightarrow 0} H_\delta^\lambda(E)$$

$\mathcal{H}^\lambda$ is a Borel measure and regular

$$\begin{aligned} \int_{\Omega} \operatorname{div} \varphi &= \int_{\mathbb{R}^N} X_n \operatorname{div} \varphi = \langle X_n, \operatorname{div} \varphi \rangle_{\mathcal{D}', \mathcal{D}} \\ &= - \langle \nabla X_n, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \end{aligned}$$

$$\text{and } 1.1 \leq \| \varphi \|_{\mathcal{D}} P(\Omega)$$

According to the Riesz representation theorem $P(\Omega) \xrightarrow{+} \rightarrow$ (positive linear form with continuous L^∞ -norm injection)

There exists a Radon measure (regular Borel measure finite on compact sets)
 $\exists \mu_n$

$$\exists D_n \in L^2(\mathbb{R}^N, d\mu_n) \text{ such that } \nabla X_n = - D_n d\mu_n$$

$$|D_n| = 1 \quad \mu_n\text{-a.e.}$$

$$\begin{aligned} \text{So we get finally } \int_{\Omega} \operatorname{div} \varphi &= \int X_n \operatorname{div} \varphi \\ &= - \langle \nabla X_n, \varphi \rangle \\ &= \int_{\mathbb{R}^N} \varphi D_n d\mu_n \end{aligned}$$

NB: $\mu \in \mathbb{R}$
 $\rightarrow |\mu| = \varepsilon_\mu$
 with $\varepsilon = \pm 1$ μ -a.e.

μ in $\mathbb{R} \rightarrow |\mu| = \pm \varepsilon_\mu$
 with $|\varepsilon| = 1$ μ -a.e.

Lebesgue - Besicovitch differentiation theorem

$$\text{Recall } f \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \frac{\int_{B(x, r)} f(y) dy}{|B(x, r)|} \xrightarrow[r \rightarrow 0^+]{\text{a.e.}} f(x)$$

$$f \in L^1_{\text{loc}}(\mathbb{R}^N, d\mu) \quad \frac{\int_{B(x, r)} f(y) d\mu(y)}{|\mu(B(x, r))|} \xrightarrow[r \rightarrow 0^+]{\mu\text{-a.e.}} f(x)$$

so we have $\frac{\int_{B(x,n)} v_n d\mu_n}{\mu_n(B(x,n))} \xrightarrow[n \rightarrow 0^+]{\text{a.e.}} v_n(x) ?$

*** Definition: $\partial^* \Omega = \{x \in \mathbb{R}^N, \mu_n(B(x,n)) > 0 \text{ for } n \text{ small}\}$

$$\lim_{n \rightarrow 0^+} \frac{\int_{B(x,n)} v_n d\mu_n}{\mu_n(B(x,n))} \text{ exists} := v_n(x)$$

$$\text{and } |v_n(x)| = 1$$

(inner No
of but measure $\neq 1$)

Property $\rightarrow \mu_n(\mathbb{R}^N \setminus \partial^* \Omega) = 0$

NB: If Ω is open, then $\partial^* \Omega \subseteq \partial \Omega$. It works also for measurable sets but we need to redefine a representative of $\partial \Omega$ (define a.e., ...)

Thm: ① $\mu_n = \frac{1}{n^{N-1}} \Big|_{\partial^* \Omega}$ or $\int_{\partial^* \Omega} v_n d\mu_n = \int_{\partial^* \Omega} \chi_{\Omega} d\mu_n$



$$\begin{aligned} x_0 &\in \partial^* \Omega \\ \parallel \\ 0 \end{aligned} \quad \Omega_x = \left\{ \begin{aligned} & \{x \in \mathbb{R}^N, x_0 + nx \in \Omega\} \\ & (= \frac{1}{n} \Omega) \end{aligned} \right\}$$

we have $X_{\Omega_n} \xrightarrow[n \rightarrow 0^+]{\text{a.e.}} X_H = \{x = (x', x_N), x_N \leq 0\}$

Remark on theorem: The theorem is optimal. For $N=2$, we have a counterexample: Simon's cone in $\mathbb{R}^2$



we have a global minimizer for the perimeter but a singularity

Quasi-minimizers $\alpha \in [0, 1]$



$B(x_0, r)$

$$x_0 \in D_\delta = \{x \in D, \alpha(x, \lambda \Omega) > \delta\}$$

$$r < \frac{\delta}{2}$$



$$\Omega \Delta \hat{\Omega} = (\hat{\Omega} \setminus \Omega) \cup (\Omega \setminus \hat{\Omega}) \subset B(x_0, r)$$

$$P(\hat{\Omega}, B(x_0, r)) \leq P(\Omega, B(x_0, r)) + C r^{N-1+\alpha}$$

Theorem: If $\hat{\Omega}$ is a quasi-minimizer α , then

(i) $\partial^* \hat{\Omega} \cap D$ is a $C^{1,\alpha}$ hypersurface

(ii) $\forall \lambda > N-1, \mathcal{H}^{\lambda}((\partial \hat{\Omega} \setminus \partial^* \hat{\Omega}) \cap D) = 0$

Question: Why is the solution of problem (i) a quasi-minimizer?

Let's define $(P_\lambda) \min \{P(\Omega) + \lambda | |\Omega| - m |, \Omega \subset D\}$

Proposition: $\exists \lambda^*, \forall \lambda \geq \lambda^*, (P_\lambda) \Leftrightarrow (P)$

NB: If $\hat{\Omega}_\lambda$ is a solution of (P_λ) such that $|\hat{\Omega}_\lambda| = m$, then we have $(P_\lambda) \Leftrightarrow (P)$

(i) Proof: $P(\hat{\Omega}_\lambda) + \lambda \underbrace{||\hat{\Omega}_\lambda| - m|}_{=0} \leq P(\Omega) + \lambda \underbrace{||\Omega| - m|}_{=0}$

(ii) $\hat{\Omega}_\lambda$ solution of (P) (problem (i)). We have:

$$P(\hat{\Omega}_\lambda) + \lambda \underbrace{||\hat{\Omega}_\lambda| - m|}_{=0} \leq P(\hat{\Omega}_\lambda) \leq P(\Omega) + \lambda ||\Omega| - m|$$

NB: If $\hat{u}$ is a solution of (1), for $\lambda \geq \lambda^*$, it is a solution for (2). Then, we may write:

$$P(\hat{u}) = P(\hat{u}) + \lambda |\hat{u}| - m \\ \leq P(u) + \lambda |u| - m$$

$$P(\hat{u}, B(m, \lambda)) \leq P(u, B(m, \lambda)) + C\lambda^N$$

so the solution of problem (1) is a quasi-minimizer

How to get more regularity?



$$y = h(x) \\ y_t = h(x) + t\psi(x) \quad \int \psi = 0$$

$$\frac{d}{dt} \int \sqrt{1 + |\nabla h + t \nabla \psi|^2} dx = 0 = \int \frac{\nabla h \cdot \nabla \psi}{\sqrt{1 + |\nabla h|^2}} = - \int \psi \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right)$$

BOOTSTRAP

$$\text{We get } \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = \text{cste} \quad \text{EXERCISE} \rightarrow \text{Do it when } N=2$$

II. OPTIMAL SETS FOR THE DIRICHLET ENERGY:

Formally, we have

D open bounded box, $\Omega \subset D$
 $m \in L^1_0(D)$
 $f \in L^\infty(D)$

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

$$J(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_\Omega|^2 = \frac{1}{2} \int_\Omega f u_\Omega$$

$$\min \{ J(u_n), n \leq D, |u_n| = m \}$$

← because of the monotony

NB: $J(u_n) = \min \{ J(u), u \in H_0^1(\Omega) \}$

If $\Omega_1 \subseteq \Omega_2$, then $H_0^1(\Omega_1) \subseteq H_0^1(\Omega_2)$ so $J(u_n) = \min \{ J(u), u \in H_0^1(D) \mid |u_n| \leq m, \{u_n\}'' \cup \{0\} \}$

Equation $\times u_n \Rightarrow \int_{\Omega} |\nabla u_n|^2 = \int_{\Omega} f u_n$

$$J(u_n) = \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 = \frac{1}{2} \int_{\Omega} f u_n$$

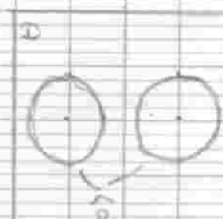
$N=2$



or small



cusp may appear



or

morally enable to write that at boundary) why? Because Euler equation is $|\nabla u_n|^2 = c \neq 0$ on $\partial \Omega$

If then 1 + etc)
impossible

Morally, if $\begin{cases} u_n > 0 \text{ regular} \\ u_n \text{ changes sign (not regular)} \end{cases}$

Mardi 05/06/12

BERND KAUFHL

11:00 - 12:30

P.S.: How to prove quasi-convexity in another way?

Verifying level sets of an allegedly non-quasi-convex function
 $u_{qc}(x) = \text{quasi-convex hull of } u$

$u_{qc}(x) \geq u(x)$

Strategy $\rightarrow$ show that $u_{qc} \leq C$ via comparison principle

1st derivative Gotsant + Salani
2nd derivative Gress + Kaufhl

Square opaque problem: what about a circle instead of a square

Fence of length 2π seems unavoidable



U-shape has length $\pi + 2$

Can you fit the line inside? YES



Lamellar have height ≤ 2
 $N \rightarrow +\infty$, they add $+2$

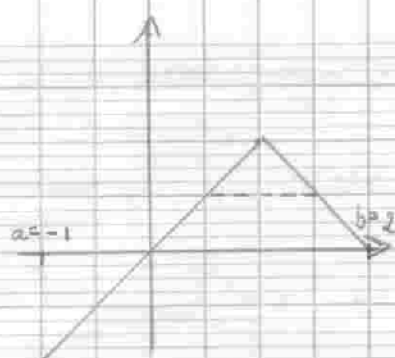
Monotone decreasing rearrangement of function

$f: [a, b] \rightarrow \mathbb{R}$ is decreasing (as non-increasing)

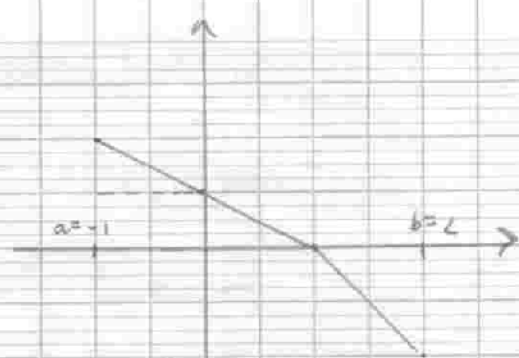
then each level set $S_t = \{x \in [a, b], f(x) \geq t\}$ is an interval with left-endpoint a .

Given a compact subset $D \subseteq [a, b]$, define $D^* = [a, \text{at}(D)]$ if $D \neq \emptyset$ and $f^*(x) = \sup\{c \in \mathbb{R}, x \in S_c\}$ is monotone decreasing and total variation $TV(f^*) \leq TV(f)$.

Example:



$$u(x) = \begin{cases} x & x \in [-1, 1] \\ 2-x & x \in [1, 2] \end{cases}$$



$$u^*(x) = \begin{cases} \frac{1}{2} + \frac{x}{2} & x \in [-1, 1] \\ 1-x & x \in [1, 2] \end{cases}$$

It suggests $\int_a^b |f| \geq \int_a^b |f^*|$

Observation ①: f^* is flatter than f

Thm: If u is Lipschitz with Lipschitz constant L , so then is u^*

Proof: (i) Only for the example
(simpler because each level set is an interval)
in which:

$$\Omega_c = [x_1(c), x_2(c)] \text{ where } \begin{cases} x_1(c) = c & c \in [-1, 1] \\ x_2(c) = 2-c & c \in [1, 2] \end{cases}$$

We have $\Omega_c^* = [-1, x^*(c)]$
with $x^*(c) = x_2(c) - x_1(c)$

u^* is decreasing by definition so $\frac{du^*}{dx} \leq 0$ and to estimate $\left\| \frac{du^*}{dx} \right\|_{L^\infty}$ it suffices to find lower bound of $\frac{du^*}{dx} (\geq -L)$

$$\frac{du^*}{dx} = \frac{1}{du^*/dc} \text{ where } \frac{dx^*}{dc} = \frac{dx_2}{dc} - \frac{dx_1}{dc} = \begin{cases} 0 & c \in [-1, 1] \\ 1 & c \in [1, 2] \end{cases}$$

$$\leq - \left(\frac{du}{dx} \right)^{-1}_{x_2} \geq - \frac{1}{L}$$

done

$$0 \geq \frac{du^*}{dx} \geq -L$$

(ii) Proof works for any piecewise affine function

(iii) If $u \in W^{1,\infty}$, approximation by piecewise linear interpolation

$$\text{So, we get } \begin{cases} \|u\|_1 = \|u^*\|_1 & \text{and we shall see later } \|u\|_p = \|u^*\|_p \\ \|u\|_{L^\infty} = \|u^*\|_{L^\infty} \end{cases} \quad \forall p \in [1, \infty]$$

Observation ②: u and u^* are equimeasured $|S(u)| = |S(u^*)|$

Observation ③: Rearrangement is order preserved

$$u \leq v \Rightarrow u^* \leq v^*$$

Observation ④: The mapping $u \rightarrow u^*$ is non-linear



Observation ⑤: Carathéodory's principle



f continuous
or maximal

$$\int_a^b f(u) dx = \int_a^b f(u^*) dx$$

Observation ⑥: Hardy-Littlewood inequality



$$\int_a^b u^* v^* \geq \int_a^b uv$$

$$\forall u, v \geq 0$$

Do it for characteristic, step, L^1 functions.

Observation ①: $\int_a^b u^* v^* \leq \int_a^b uv \quad \forall u, v \geq 0$

v^* is monotone increasing rearrangement of v

Observation ②: The mapping $u \mapsto u^*$ is non-expansive in L^2

$$\forall u, v \geq 0, \quad \int_a^b |u^* - v^*|^2 \leq \int_a^b |u - v|^2 = \int_a^b u^2 - 2 \int_a^b uv + \int_a^b v^2$$

$$\int_a^b u^2 \geq -2 \int_a^b uv \quad \int_a^b v^2$$

One can even show that $\int_a^b j(u^* - v^*) \leq \int_a^b j(u - v)$ for any $j: [0, +\infty] \rightarrow [0, +\infty]$ convex $j(0) = 0$

This means that in particular $u \mapsto u^*$ maps L^1 into L^1

GOAL: Pólya-Szegő inequality $\rightarrow \|u^*\|_{L^1} \leq \|u\|_{L^1}$

Definitions: $\bar{u} = \bar{u}^*$ $u: \bar{u} \rightarrow \mathbb{R}$

u is simple if u is piecewise linear

smooth if $u \in C^1(\bar{u})$ and if (i) $\{x \in \bar{u}, u(x) = c\}$ is finite $\forall c \in \{\min u, \max u\}$
(ii) $\{x \in \bar{u}, \frac{du}{dx} = 0\}$ is finite

A simple function is min if $\frac{du}{dx} \neq 0$ a.e.

Proposition: If u is simple and smooth, $F: \mathbb{R} \rightarrow \mathbb{R}^+$ continuous & non decreasing, convex, then

$$(i) \int_{\mathbb{R}} F(u) G(|\frac{du}{dx}|) \geq \int_{\mathbb{R}} F(u^*) G(|\frac{du^*}{dx}|)$$

(ii) Moreover, if F is positive, & strictly increasing or strictly convex, then equality holds if and only if u is already monotone.

Proof of slope-Young inequality:

First, we prove it for a dense subset of $W^{1,p}$ (piecewise linear function)
 then for any $u \in W^{1,p}$ (by interpolation)
 then $u \mapsto u^*$ maps $W^{1,p}$ into $W^{1,p}$



Each D_i decompose into subintervals x_{ij} with $j=1, \dots, N(i)$ (counted from left to right, in which u is linear)

For every height $\lambda \in (a_i, a_{i+1})$ and $j=1, \dots, N(i)$ let $p_j(\lambda)$ be the unique value of x in x_{ij} in which $u(x) = \lambda$

u^* is monotonically decreasing, so $p^*(\lambda)$ is the unique value of x for which $u^*(x) = \lambda \in (a_i, a_{i+1})$

Without loss of generality, let $\Omega = [0, w]$ and let u to attain its maximum at 0 . Then $\text{sign } \frac{du}{dx}(p_j) = -1$

$$\text{sign } \frac{du}{dx}(p_j) = (-1)^j$$

$$p^*(h) = \begin{cases} \sum_{i=1}^{N(h)} (-1)^{j_i} p_i(h) & \text{if } N \text{ is odd} \\ \sum_{j=1}^{N(h)} (-1)^{j_i} p_i(h) + w & \text{if } N \text{ is even} \end{cases} \quad (p_1 = p_0 + w \text{ in our example})$$

$$\frac{du}{dx} = \frac{1}{dp_i/dh} \quad \text{in } D_{ij}$$

$$\frac{du^*}{dx} = \frac{1}{q^*/dh}$$

GOAL: Prove the inequality

$$\int F(u) G\left(\left|\frac{du}{dx}\right|\right) \geq \int F(u^*) G\left(\left|\frac{du^*}{dx}\right|\right)$$

We want to show $\int_{D_i} F(u) G\left(\left|\frac{du}{dx}\right|\right) \geq \int_{D_i^*} F(u^*) G\left(\left|\frac{du^*}{dx}\right|\right)$ and we do so by doing it $\forall D_i$

On D_i , we introduce h as an independent variable

$$\begin{aligned} \sum_{j=1}^{N(h)} \int_{D_{ij}} F(u) G\left(\left|\frac{du}{dx}\right|\right) dx &= \sum_{j=1}^{N(h)} \int_{a_i}^{a_{i+1}} F(h) G\left(\left|\frac{dp_i}{dh}\right|^{-1}\right) \left|\frac{dp_i}{dh}\right| dh \\ &\geq \int_{a_i}^{a_{i+1}} F(h) G\left(\left|\frac{dp^*}{dh}\right|^{-1}\right) \left|\frac{dp^*}{dh}\right| dh \\ &\quad \cdot \left| \sum_{j=1}^{N(h)} (-1)^{j_i} \frac{dp_j}{dh} \right|^{-1} \end{aligned}$$

NB: N even or odd are now both included (does not make any difference)

The sign of $\frac{dp_i}{dh}$ alternate so $\left| \sum_{j=1}^{N(h)} (-1)^{j_i} \frac{dp_j}{dh} \right| = \sum_{j=1}^{N(h)} \left| \frac{dp_j}{dh} \right|$

Set $\alpha_j = \frac{|dp_j/dh|}{\sum_{j=1}^{N(h)} |dp_j/dh|}$. We have $\sum_{j=1}^{N(h)} \alpha_j = 1$

By convexity of G , we have:

$$\begin{aligned} \sum_{j=1}^{N(\lambda)} \alpha_j G\left(\left|\frac{dp_j}{d\lambda}\right|^{-1}\right) &\geq G\left(\sum_{j=1}^{N(\lambda)} \alpha_j \left|\frac{dp_j}{d\lambda}\right|^{-1}\right) \\ &= G\left(N \sum_{j=1}^{N(\lambda)} \left|\frac{dp_j}{d\lambda}\right|^{-1}\right) \\ &\geq G\left(\sum_{j=1}^{N(\lambda)} \left|\frac{dp_j}{d\lambda}\right|^{-1}\right) \\ &= G\left(\sum_{j=1}^{N(\lambda)} \left|\frac{dp_j}{d\lambda}\right|^{-1}\right) \end{aligned}$$

This proves the inequality (i) of the proposition (i) F must be ≥ 0 and (ii) if G is strictly monotone.

Question: What about symmetrically decreasing rearrangement?

(Consider a triangle, base length 2, height 1, peak at $(1,1)$)



Claim $\Rightarrow$ the isosceles triangle with peak at 0 has smaller perimeter

Perimeter $= 2 + \int_{-1}^1 \sqrt{1 + |u'|^2} dx$ and $g(a) = 2 + \sqrt{1 + a^2} + \sqrt{1 + (1-a)^2}$ is convex

NB: The area does not change

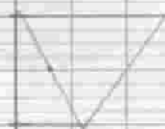
Corollary 1: Among all triangles of given area, only the equilateral minimizes the perimeter.

Corollary ②: Among all quadrilaterals of given area, only the square minimizes the perimeter.

Proof: ① if it is not a convex quadrilateral, then convexify and better



② Symmetrize with respect to a diagonal to arrive at symmetric rhombus



③ Then pass to a rectangle then to a square



Corollary ③: Among all N -gons of given area, regular ones minimize the perimeter

Proof: ① Convexify, introduce artificial corners if necessary

② Adjacent sides have equal length so the polygon is equilateral



③ Show that adjacent angles must be equal by studying perimeter of a trapezoid with 3 equally long sides



Mardi 05/01/2

BRUN BLUR

15:30 - 17h00

Definition: D bounded open set, $E \subset D$

The capacity of E in D

$$\text{cap}(E, D) = \inf \left\{ \int_D |\nabla u|^2 \mid u \in H^1_0(D), u \geq 1 \text{ a.e. in a neighborhood of } E \right\}$$

Alternative definition: $\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \mid u \in H^1(\mathbb{R}^N), u \geq 1 \text{ a.e. in a neighborhood of } E \right\}$

If u satisfies $\text{cap}(E, D) = \int_D |\nabla u|^2$ then u is a solution of

$$\begin{cases} -\Delta u = 0 & \text{in } D \setminus E \\ u = 1 & \text{on } E \\ u = 0 & \text{on } \partial D \end{cases}$$

Examples: * 1 point in $\mathbb{R}$



* 1 point in $\mathbb{R}^2$



$$\text{cap}(0, B_1) = 0$$

* 1 segment in B_1



$$\int_B u^2 \leq C \int_D |\nabla u|^2 \quad \text{trace theorem}$$

conclusion: capacity of a segment > 0 whereas $\lambda(\text{segment}) = 0$

NB: If a set can be seen through its trace, then capacity > 0

capacity = maximize the defect of continuity of a Sobolev function
 Lebesgue = maximize the defect of continuity of a L^1 -function

If K is closed with $\text{cap}(K, D) = 0$, then for $\Omega \subseteq D$ we have

$$H_0^1(\Omega) = H_0^1(\Omega \setminus K)$$

$$C_c^\infty(\Omega)^{\|\cdot\|_{H_0^1}} = C_c^\infty(\Omega \setminus K)^{\|\cdot\|_{H_0^1}}$$

conclusion: If one compute the record of a circle, idem circle - center

$$\text{Cap}(E \cup E_2, D) \leq \text{Cap}(E_1, D) + \text{Cap}(E_2, D) \quad \text{the capacity is not a measure, it is subadditive.}$$

Definitions * We say that a property holds quasi-everywhere (q.e.) if it holds in the complement of a set of zero capacity.

** $f: D \rightarrow \mathbb{R}$ is quasi-continuous if $\forall \varepsilon > 0, \exists$ open set U with $\text{Cap}(U, D) < \varepsilon$ and such that $f|_{D \setminus U}: D \setminus U \rightarrow \mathbb{R}$ is continuous.

Ideas: w.i.p. Sobolev functions are more regular than L^p functions. They enjoy better continuity - quasi-continuity. We can use better pointwise properties.

Theorem: Let $u \in H_0^1(D)$. Then the limit $\lim_{r \rightarrow 0} u$ exists q.e. in D and $\tilde{u}(x) = \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} u$ is a quasi-continuous function.

NB: For L^p -functions, the limit exists a.e. and is a.e. continuous. Here, we have more:

$$\boxed{2-D} \quad \|\cdot\|_{Q^p \cap B(0,1)}$$

Definition: A set $\Omega \subseteq \mathbb{D}$ is called quasi-open if $\forall \varepsilon > 0$
 $\exists V$ open such that $\text{cap}(V, \mathbb{D}) < \varepsilon$ and $\Omega \cup V$ is open.

NB: If $u \in H_0^1(\mathbb{D})$ and u is quasi-continuous, then $\{u \geq c\}$ is quasi-open.

Definition: Let $\Omega \subseteq \mathbb{D}$ be a quasi-open set. By definition

$$H_0^1(\Omega) = \{u \in H_0^1(\mathbb{D}), u = 0 \text{ q.e. } ^c \Omega\}$$

Then $H_0^1(\Omega)$ is closed. This is a consequence of the property
 $u_n \in H_0^1(\mathbb{D}) \Rightarrow u_n(x) \xrightarrow{\text{q.e.}} u(x)$
 $u_n \xrightarrow{H_0^1(\mathbb{D})} u$

Open question: If this holds if $u_n \rightarrow u$?

Theorem: (Hedberg) If Ω is open, then we have
 $H_0^1(\Omega) = \{u \in H_0^1(\mathbb{D}), u = 0 \text{ q.e. } ^c \Omega\}$

Example: $u_n \xrightarrow{H_0^1(\mathbb{D})} u$ and $u = 0$ at Ω
 $\Downarrow ?$
 $u \in H_0^1(\Omega)$

Let $f \in L^2(\mathbb{D})$, Ω a quasi-open set included in $\mathbb{D}$

$$\begin{aligned} \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} &\iff \begin{cases} u \in H_0^1(\Omega) \\ \forall \varphi \in H_0^1(\Omega), \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \end{cases} \\ &\iff u \text{ minimizes } \min_{v \in H_0^1(\Omega)} \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \end{aligned}$$

Theorem: (shape stability theorem) Let (Ω_k) , n be open & quasi-open subsets of $\mathbb{R}^n$. The following assertions are equivalent:

($H^1(\Omega)$ is the same)
 def: Ω means $\Omega_k \rightarrow \Omega$

① $\forall f \in L^2(\Omega), u_{\Omega_k, f} \xrightarrow{H^1(\Omega)} u_{\Omega, f}$

② Only for $f \equiv 1, u_{\Omega_k, 1} \xrightarrow{H^1(\Omega)} u_{\Omega, 1}$

③ $H_0^1(\Omega_k) \xrightarrow{\text{norm}} H_0^1(\Omega)$

④ $F_k: L^2(\Omega) \rightarrow \{R \cup \{+\infty\}\}$
 $\omega \mapsto F_k(\omega) = \begin{cases} \int_{\Omega} |\nabla u|^2 & \text{if } u \in H_0^1(\Omega_k) \\ +\infty & \text{if not} \end{cases}$

and $F_k \xrightarrow{F-\text{ot}} F_\Omega$

$\left(u_{\Omega_k, f} \text{ solves } -\Delta u = f \text{ in } \Omega_k \right)$
 $u=0 \text{ on } \partial\Omega_k$

1) $\forall u \in H_0^1(\Omega), \exists v_k \in H_0^1(\Omega_k), v_k \xrightarrow{H^1(\Omega)} u$ strong
 2) $\forall v \in H_0^1(\Omega_k)$ such that $v_k \xrightarrow{H^1(\Omega)} v \Rightarrow v \in H_0^1(\Omega)$

1) $\forall v_k \xrightarrow{L^2(\Omega)} v, F_k(\omega) \leq \liminf F_k(\omega)$
 2) $\forall v \in L^2(\Omega), \exists v_k \xrightarrow{L^2(\Omega)} v, F_k(\omega) \geq \limsup F_k(\omega)$

Remarks: * The resolvent operator $R_{\Omega_k}: L^2(\Omega) \rightarrow L^2(\Omega)$
 $f \mapsto R_{\Omega_k}(f) = u_{\Omega_k, f}$

Property ① says pointwise $\forall$ of operator R

$\omega \Rightarrow R_{\Omega_k}(f) \xrightarrow{L^2(\Omega)} R_\Omega(f)$

$\Downarrow$
 $R_{\Omega_k} \xrightarrow{\mathcal{L}(L^2(\Omega))} R_\Omega$

consequence of compact injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$

$\Downarrow$
 $\frac{1}{\lambda_1(\Omega_k)} - \frac{1}{\lambda_1(\Omega)} \leq \|R_{\Omega_k} - R_\Omega\|_{\mathcal{L}(L^2(\Omega))}$

* Property ③ does not depend of the operator (Δ)

$$\begin{cases} -\operatorname{div}(A \nabla u) = f & H_0^1(\Omega) \\ u = 0 \end{cases}$$

If H_f is δ -CV in $L^2(\Omega)$ H and v_2 are minimizers of H_f such that $v_2 \xrightarrow{w} v$, then v is a minimizer of H .

* If G is continuous $G: L^2(\Omega) \rightarrow \mathbb{R}$, then $F + G \xrightarrow{L^2(\Omega)} F + G$
 $G(u) = -\frac{1}{2} \int_{\Omega} f u$

Under weak assumptions, the Γ -CV is compact!

Proof:

② $\Rightarrow$ ① We can assume by a density argument that $f \in L^2(\Omega)$
 (Assume Ω is open) $f \geq 0$

$$u_{n,1} \xrightarrow{H_0^1(\Omega)} u_{n,1} \Rightarrow u_{n,1} \xrightarrow{H_0^1(\Omega)} u_{n,1}$$

$$0 \leq u_{n,1} \leq \|f\|_{L^2} \cdot u_{n,1}$$

$$0 \leq u \leq \|f\|_{L^2} \cdot u_{n,1} \Rightarrow u \in H_0^1(\Omega)$$

$\varphi \in C_c^0(\Omega)$, $\exists \delta > 0$, $0 \leq \varphi^+ \leq C u_{n,1}$ strong maximum principle

$$\begin{cases} \varphi_2 = \min(\varphi^+, C u_{n,1}) \in H_0^1(\Omega) \\ \varphi_2 \xrightarrow{H_0^1(\Omega)} \varphi^+ \end{cases}$$

Then we have $\int_D \nabla u \cdot \nabla \varphi \stackrel{\text{weakly}}{=} \int_D f \varphi \stackrel{\text{strongly}}$

$$\int_D \nabla u \cdot \nabla \varphi^+ = \int_D f \varphi^+$$

Remark: Assume that $\Omega \xrightarrow{H^c} \Omega_0$ (in general no)

Take Ω open $\subset \mathbb{D}$ $\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$



$x_0 \in \partial\Omega, u \in C^0(\bar{\Omega}) \quad \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) = 0$

Wiener's criterion: ∞ if and only if $\int_0^1 \frac{\text{Cap}(\partial \cap B(x_0, t), B(x_0, 2t))}{\text{Cap}(B(x_0, t), B(x_0, 2t))} \frac{dt}{t} > \infty$

Example: In $\mathbb{R}^2$ with Ω simply connected, this is always true



$$\int_0^1 \frac{0}{t} dt \geq \int_0^1 \text{straight line}$$

Monday 6/6/12

BERND KAWOHL

9:00 - 10:30

Symmetric decreasing rearrangement

Converts level sets into symmetric sets:

$$\Omega_c^* = \left[-\frac{1}{2} |\Omega_c|, \frac{1}{2} |\Omega_c| \right]$$

(recall triangle)

and this rearrangement the convex functional $\int |\nabla u|^p dx$ can decrease $\|u'\|_{L^1}$

Goal $\rightarrow$ Back on the same properties than last time but with a symmetric rearrangement.

$$\int |\nabla u^*|^p \leq \int |\nabla u|^p$$

Application ①: Assume $u: [-w, w]$ is a global minimizer of

$$E(w) = \int_{-w}^w \left[\frac{1}{2} |u'|^2 + w(u) \right] dx$$

in the class of periodic function in $W^{1,2}([-w, w])$ with mean value zero

Then, the sets $\{x \in [-w, w], u(x) > 0\}$ and $\{x \in [-w, w], u(x) < 0\}$ are simply connected (modulo shift) and u is symmetrically decreasing (resp. increasing) in each of them

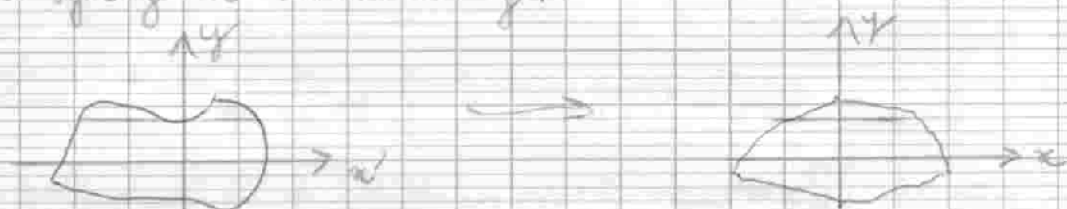
Application ②: Isoperimetric inequality HYP: $\exists$ of minimizers

connected + convex

(convex hull + rosalia)

Semir symmetrisation in direction y : For $D \subseteq \mathbb{R}^N$ and $x = (x_1, \dots, x_N)$
Set $x' = (x'_1, y)$ with $x'_1 \leq (x_1, \dots, x_{N-1})$
 $y = x_N$

For fixed $x' \in \mathbb{R}^{N-1}$, consider the 1-D set $\{y \in \mathbb{R} \mid (x', y) \in D\}$ and symmetrize it in direction y .



Property: $|D| = |D^*|$

Steiner symmetrization decreases the perimeter of a convex set of a set with smooth boundaries in $\mathbb{R}^2, \mathbb{R}^3, \dots$ of a Caccioppoli set (Fusco, Giacchi, Maggi, Frasca)

If Ω is a level set of a function $u(x, y)$ we can define Ω^* and u^* as before, provided Ω is bounded.

For simplicity, let's assume:

(A) $\Omega \subseteq \mathbb{R}^N$ is bounded and $u \geq 0$ in Ω , $u = 0$ on $\partial\Omega$.

(A') $\Omega = \Omega' \times [-w, w]$ and u is periodic with respect to y .

Definition: $u: \Omega \rightarrow \mathbb{R}$ is simple if u is piecewise linear and continuous

min if simple and $\frac{\partial u}{\partial y} \neq 0$ a.e. in Ω

smooth if $u \in C^1(\Omega)$ and $\forall a \in \mathbb{R}^{N-1}, a \in \mathbb{R}$,

$$\begin{cases} u(a', y) = \lambda & \text{for finitely many } y \\ u_y(a', y) = 0 & \text{for finitely many } y \end{cases}$$

Theorem: If u is nice or smooth, $F \geq 0$, G non-increasing and convex, then we have

$$\int_{\Omega} F(u) G(|\nabla u|) dx dy \geq \int_{\Omega} F(u^*) G(|\nabla u^*|) dx dy$$

If moreover $F > 0$, G strictly non-increasing and convex, then equality holds if and only if $u = u^*$ (modulo translation)

Corollary: $u \in W^{1,p}(\Omega) \Rightarrow u^* \in W^{1,p}(\Omega^*)$
and we have:

$$\|u^*\|_{W^{1,p}} \leq \|u\|_{W^{1,p}}$$

Application: Faber-Krahn

$$\lambda_p(\Omega) = \inf_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}$$

Then, a minimizer u_p solves $\begin{cases} \Delta_p u + \lambda_p(\Omega) |u|^{p-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

and $u_p \geq 0$ in Ω , so it suffices to minimize $R(u) = \frac{\|\nabla u\|_p^p}{\|u\|_p^p}$ over non-negative functions

$$\lambda_p(\Omega) = R(u_p, \Omega) \geq R(u_p^*, \Omega^*) \geq \inf R(u, \Omega^*) = \lambda_p(\Omega^*)$$

So Steiner symmetrization decreases the λ_p .

Corollary: Among all clamped membranes of given area, the circular disc has the lower fundamental frequency (Faber-Krahn)

The Schwarz representation: level sets are transformed into concentric balls.

Recent papers by Fusco, Giacchi, Maggi, Fratelli decrease $\lambda_p(u) - \lambda_p(u^*)$ in terms of measures of non-symmetry like $\frac{|\partial u|^{p-1}}{|\partial u|}$ and provide qualitative inequalities.

Classical result: Among all elastic bars of unit section of given area, the tube has the less tension.

Circular symmetrisation: sets are pushed to the north pole on circular arcs on spherical caps.



If we represent D in polar coordinates, we Steiner-symmetrize in direction φ .

$$\int |\nabla u|^2 = \int \left(u_\rho^2 + \frac{1}{\rho^2} \sum u_\theta^2 \right) \rho d\rho d\theta$$

This can be generalized $\int \left(x_1(u) \left| \frac{\partial u}{\partial x_1} \right|^2 + x_2(u) \left| \frac{\partial u}{\partial x_2} \right|^2 \right) F(x, u) dx$ (cf. th 2.32 in the lecture notes)

Back to the isoperimetric conjecture:

$$\lambda_2 = \inf_{S \neq \emptyset} \frac{\int |\nabla u|^2}{\int u^2}$$

If I have any domain



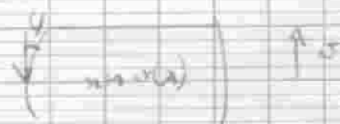
have to work but $\int |\nabla u|^2$ does not decrease

Star-shaped rearrangement is weighted rearrangement



is either equimeasurable but then the Polya-Szegő inequality fails or Polya-Szegő works but not equimeasurable

Newton resistance of a boat:



$$\text{resistance} = \frac{1}{\sqrt{1 + |v|^2}}$$

$$R(v) = \int_{\mathbb{R}} \frac{dx}{\sqrt{1 + |v|^2}}$$

Mercredi 6/6/12 // (note) OPTIMAL SETS FOR THE DIRICHLET ENERGY:

NICOLAS PIERRE

11.00 - 12.30

$D \subseteq \mathbb{R}^N$ open bounded set $f \in L^q(D)$ given
 $m \in (0, |D|)$

$$(J) \quad \begin{cases} J(u) = \min \{ J(v), v \in H_0^1(D), |Dv| \leq m \} \\ |Dv| \leq m \end{cases}$$

$\uparrow$ since $|v| \neq 0$

$$J(u) = \frac{1}{2} \int_D |\nabla u|^2 - \int_D f u$$

This is the set version of the functional problem $\min \{ J(u, v) \mid u \in H_0^1(D), v \in D, |Dv| \leq m \}$
 with u solution of $\begin{cases} -\Delta u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$
 because $J(u) = \min \{ J(v), v \in H_0^1(D) \}$

NB: If $|A| < m$ and if $\begin{cases} \Omega \subseteq \hat{\Omega} \\ |\hat{\Omega}| = m \\ \hat{\Omega} \subseteq D \end{cases}$ then $J(u_{\hat{\Omega}}) \leq J(u_{\Omega})$

because $\Omega \rightarrow \hat{\Omega}$ is decreasing (reason $\rightarrow H^1_0(\Omega) \subseteq H^1_0(\hat{\Omega})$)

Theorem: $\exists \hat{\Omega}$ quasi-open such that $\hat{\Omega} \subseteq D$, $|\hat{\Omega}| = m$ and $J(u_{\hat{\Omega}}) = \min\{J(u_{\Omega}), \Omega \subseteq D, |A| = m\}$

Remark: In general, it is quasi-open. One may have $f \in H^{-1}(\Omega)$ such that $\hat{\Omega}$ is quasi-open and not open.

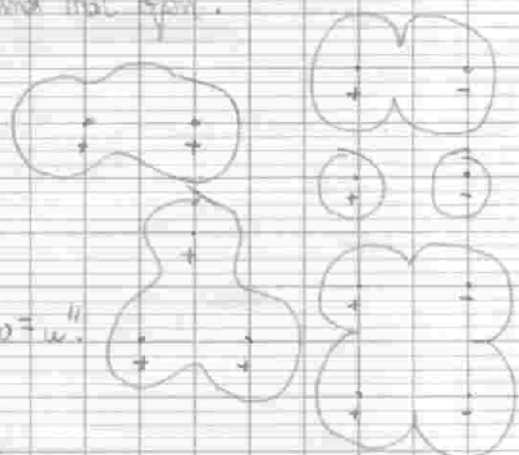
A "pathological" situation

let $w \in C_c(\Omega)$ such that $|A_w| = m$

let $f = -\Delta w$ so that $w \in H^1_0(\Omega)$

$$-\Delta w = f$$

and w is a solution of our problem with " $w = u$ ".



Some remarks is valid if $|A_w| < m$

conclusion: Only "situation" of the constraint $|A_w| \leq m$ will allow regularity of the optimal shape Ω_u .

($\Leftrightarrow \exists$ Lagrange multiplier $\neq 0$)

$$\forall w \in H^1_0(\Omega) \quad (J_{\lambda}) \quad J(u_u) + \lambda (|A_{u_u}| - m)^+ \leq J(w) + \lambda (|A_w| - m)^+$$

Proposition: $\exists \lambda^*$ such that $\forall \lambda \geq \lambda^*$, $(J) \Leftrightarrow (J_{\lambda})$

NB: If u_N is solution of (P_N) such that $|Du_N| \leq m$ then $(J) \Leftrightarrow (P_N)$

$| \leq 1$ or

$$\Leftrightarrow \text{let } u \in (J) \quad J(u) \leq J(u_N) \leq J(u) + \lambda(|Du_N| - m)^+ \quad \textcircled{a}$$

$\begin{array}{c} \text{since} \\ |Du_N| \leq m \\ \downarrow \\ J(u) + \lambda(|Du_N| - m)^+ \end{array}$

Proof of proposition: If $|Du_N| > m$ then necessarily $\lambda \leq \lambda^*(N, m)$ to be proved later

1st proof: Works for D starshaped $tD \subseteq D \quad \forall t \in (0, 1)$

$$u_t = u_N\left(\frac{x}{t}\right) \quad \begin{array}{l} t \leq 1 \\ t \geq 1 \end{array} \quad \begin{array}{l} Du_t = t^{-N} Du_N \\ |Du_t| = t^{-N} |Du_N| \end{array}$$

$$J(u_N) + \lambda(|Du_N| - m)^+ \stackrel{\text{HYP}}{\leq} J(u_t) + \lambda(|Du_t| - m)^+ \quad \text{if } t \text{ close enough to 1}$$

$$t \rightarrow 1 \quad \lambda |Du_N| (1 - t^N) \leq \frac{J(u_t) - J(u_N)}{1 - t}$$

$$\lambda N m \leq \lambda N |Du_N| \leq - \frac{d}{dt} J(u_t) \bigg|_{t=1}$$

$$J(u_t) = \frac{1}{2} \int_D \frac{1}{t^2} |Du_N\left(\frac{x}{t}\right)|^2 dx - \int_D u_N\left(\frac{x}{t}\right) f(x) dx$$

Change of variable $\rightarrow y = \frac{x}{t} \quad dx = t^N dy$

$$J(u_t) = \frac{t^{N-2}}{2} \int_D |Du_N|^2 - \int_D u_N\left(\frac{x}{t}\right) f(x) dx$$

Then, we get $\lambda |u| \leq \dots \leq -\frac{d}{dt} \int_{\mathbb{R}^N} u^2$

$N \geq 2$

$$= -\left[\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} (|\nabla u \cdot x|) f(u) dx \right]$$

But we have

$$\int_{\mathbb{R}^N} (|\nabla u \cdot x|) f(u) dx \leq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + C_N \int_{\mathbb{R}^N} (f(u) \cdot u)^2 dx$$

We get finally

$$\lambda \leq \frac{C_N \int_{\mathbb{R}^N} (f(u) \cdot u)^2}{m(N)} := \lambda^*$$

2nd proof

$$u_t(x) = (u(x-t))_+ - (u(x+t))_-$$

$$\begin{pmatrix} \nabla u_t = \nabla u^+ X_{|x|>t} \\ + \nabla u^- X_{|x|>t} \end{pmatrix}$$

Same argument as the 1st proof

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} f(u) + \lambda (|u| - |u_t|) \leq \frac{1}{2} \int_{|x|>t} |\nabla u|^2 - \int_{\mathbb{R}^N} f(u)$$

$$\frac{1}{2} \int_{|x|>t} |\nabla u|^2 + \lambda |u| - |u_t| \leq \int_{\mathbb{R}^N} f(u - u_t)$$

$$\leq t \int_{\mathbb{R}^N} |f|$$

$$I_\lambda = \int_{|x|>t} |\nabla u| \left(\frac{|\nabla u|}{2} + \frac{\lambda}{|f(u)|} \right)$$

$$g(u) = \frac{u}{2} + \frac{1}{\lambda} \quad g'(u) = \frac{1}{2} - \frac{1}{\lambda u}$$

$$g(u) \geq \sqrt{2\lambda} \quad \text{so we get} \quad I_\lambda \geq \int_{|x|>t} \sqrt{2\lambda} |\nabla u|$$

u must be Lipschitz here

$$\sqrt{2\lambda} \int_0^t ds P(|u| > s) = \sqrt{2\lambda} \int_0^t ds \int_{|u|=s} d\mu^{N-1}$$

so we get $I_\lambda \geq \int_0^t ds |u_{\lambda,1}| > \lambda / \frac{N-1}{N} C_0$

And finally $t \int_D |f|^2 > I_\lambda \geq \sqrt{2\lambda} t m^{\frac{N-1}{N}} C_0$

Remarks: The same proof applies to other situations
 $\int_D |u_{\lambda,1}|^2 \rightarrow \int_D |u_{\lambda,1}|^p$

*** Optimal eigenvalue problem $\lambda(\Omega) = \min \{ \lambda(\Omega), \Omega \in \mathcal{D}, |\Omega| \leq m \}$
 $\int_D |\nabla u_\lambda|^2 = \min \{ \int_D |\nabla u|^2, u \in H_0^1(D), \int_D u^2 = 1, |u_{\text{sup}}| = m \}$

conclusion of the previous step: $\forall \lambda \geq \lambda^*$ if u is solution of (D) then $\forall u \in H_0^1(D)$,

$$J(u) \leq J(u) + \lambda (|u_{\text{sup}}| - m)^+$$

$$\text{so } \exists \lambda^* \forall \lambda \geq \lambda^*, \int_D |\nabla u_\lambda|^2 \leq \int_D |\nabla u|^2 + \lambda(u^2) + \lambda(|u_{\text{sup}}| - m)^+$$

*** $\lambda_k(\Omega) = \min \{ \lambda_k(\Omega), \Omega \in \mathcal{D}, |\Omega| = m \}$

$$\begin{aligned} \int_D |\nabla u_\lambda|^2 &\leq \int_D |\nabla u|^2 + \lambda(u^2) + \lambda(|u_{\text{sup}}| - m)^+ \\ &\quad + \lambda(\Omega) \sum_{1 \leq i \leq k-1} \int_D (u_{\text{sup}}(f_i))^2 \end{aligned}$$

orthogonal basis of
lower eigenfunctions for
 $\lambda_i \quad i=1, \dots, k-1$

Let's come back now to the Dirichlet energy:

$$\begin{cases} J(u) \leq J(v) + \lambda (|Du| - m)^+ & \forall v \in H^1_0(D) \\ |Du| = m \end{cases}$$

Question: What is the regularity of Du and of u ?

Theorem: (i) u is locally Lipschitz in D

(ii) $f \geq 0, f \not\equiv 0$ (we avoid the situation $-\Delta u = f \Rightarrow u \equiv 0$ in D)

- Du has locally finite perimeter

- $\chi^* Du$ is regular (at least $C^{1,\alpha}$) ($N \geq 2$)

and more if f is more regular (even analytic if $f = 0$ near boundary)
supp $f \subset Du$

$$H^{N-1}((Du) \setminus \chi^* Du) \cap D = \emptyset$$

For $N=2$, the full boundary is regular.

Open questions:

$$\text{Impose } H^{N-1}_p((Du) \setminus \chi^* Du) \cap D = \emptyset$$

$$\min \int_D |\nabla u|^2 + \lambda |Du|, \quad u = u_0 \text{ at } \partial D \quad \text{Alt - Caffarelli}$$

Full regularity for $N=3, u_0 \geq 0$

De Silva - Jensen: $\exists$ of a minimal cone in $\mathbb{R}^2$ (Adams)

What about the case when u changes sign?



$N=2$



(i) Half u is locally Lipschitz in $\mathbb{D}$

(ii) $\Delta u(x_0) \neq 0$

To prove Lipschitz continuity

Lemma: (i) u is continuous

(ii) Δu^+ , Δu^- are measures

$\Delta u^+ + f \chi_{u>0} \geq 0$ (positive distribution $\Rightarrow$ measure)

$x_0 \in \partial \mathbb{D}$
 $u(x_0) = 0$

$|\Delta u|(\mathbb{B}(x_0, r)) \leq \int_{\mathbb{B}(x_0, r)} r^{N-1} |f|$ this is local

* if $u \geq 0$, "easy"

* if u changes sign, monotonic formulae of Caffarelli

Let $u \in H^1(\mathbb{B}(x_0, r_0))$ and continuous with $u(x_0) = 0$

$-\Delta u^+ \leq a$ $-\Delta u^- \leq a$

$\Phi(r) = \left(\frac{1}{r^2} \int_{\mathbb{B}(x_0, r)} \frac{|\Delta u^+|^2}{|x|^{N-2}} \right) \cdot \left(\frac{1}{r^2} \int_{\mathbb{B}(x_0, r)} \frac{|\Delta u^-|^2}{|x|^{N-2}} \right)$

if $a=0$, then $r \mapsto \Phi(r)$ is monotone $\uparrow$

if $a > 0$, for $[0, \frac{1}{2}]$, $\Phi(r) \leq C(1 + \int_{\mathbb{B}(x_0, r)} u^2)$

Jeu. 07/06/12
DREIN BURK
9:00 - 10:30



$$\left. \begin{array}{l} -\Delta u = 1 \\ u \in H_0^1(\Omega) \end{array} \right\}$$

$$\Omega \text{ open } \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} u(x) = 0 \iff \int_0^R \frac{(\sup_{\Omega \cap B(x_0, t)} u) \, dt}{(\sup_{\Omega \cap B(x_0, t)} u)} = +\infty$$

Wiener
criterion

NB: In 2-D, it must be simply connected

Theorem: let $g \in H^1(\Omega) \cap C^0(\bar{\Omega})$ and h the solution of

$$\begin{cases} -\Delta h = 0 \text{ in } \Omega \\ h - g \in H_0^1(\Omega) \end{cases}$$

Then $\forall x_0 \in \Omega$, and $\forall 0 < r \leq R$, we have

$$\text{osc}(h, \Omega \cap B(x_0, r)) \leq \text{osc}(g, \Omega \cap B(x_0, R)) + \text{osc}(g, \partial\Omega) e^{-C}$$

where C depends only on the dimension

$$\text{osc}(h, A) = \left| \sup_{x \in A} h(x) - \inf_{x \in A} h(x) \right| \quad (\text{it comes from potential theory})$$

Application: Ω open $\subset \mathbb{D}$

$w_D = \chi_{\Omega, 1}$ (solution of Laplacian-Dirichlet with $f \equiv 1$)
 $w_D \in C^\infty$ in a neighbourhood of Ω

$$h = w_D - \chi_{\Omega, 1} \implies -\Delta h = 0 \text{ in } \Omega$$

$$h - g \in H_0^1(\Omega) \text{ for } g = \chi_{\Omega, 1} \in C^0(\bar{\Omega})$$

We get $\text{osc}(w_D - \chi_{\Omega, 1}, B(x_0, r)) \leq \text{osc}(w_D, \Omega \cap B(x_0, R))$

$$+ \underbrace{\text{osc}(w_D, \partial\Omega)}_{\leq \|w_D\|_\infty} e^{-C(\text{Wiener criterion})}$$

For some $s < 1$, $R = r^s$

work in 1D-2D
 simply connected or
 component has finite
 number of connected
 components

If Ω has a capacity density condition:



$$\exists C, r_0, \forall x \in \Omega, \forall r \in (0, r_0) \\ \frac{\text{Cap}(B(x, r) \cap \Omega, \mathbb{R}^n, \nu)}{\text{Cap}(B(x, r), \mathbb{R}^n, \nu)} \geq C$$

Theorem: If Ω satisfies a (C, r_0) -capacity condition
 then $\exists \alpha > 0$ such that $w_R \in C^{\alpha, \alpha}(\bar{\Omega})$
 where

$$\|w_R\|_{C^{\alpha, \alpha}} \leq C(\text{diam} \Omega, N, C, r_0)$$



Remarks: Let (Ω_k) gen sets in $\mathbb{D}$, $\Omega_k \xrightarrow{H^c} \Omega$
 Let $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\exists E_k, \forall E \subset E_k, K \subset \Omega_k \Rightarrow \varphi \in H_0^1(\Omega_k)$$

* If (Ω_k) satisfy a (C, r_0) -capacity condition

$$\Rightarrow (w_{\Omega_k}) \in C^{\alpha, \alpha} \text{ with controlled norm}$$

we identify a
 gaussian situation
 where it is true

$$\text{so } w_{\Omega_k} \xrightarrow{C^{1,2}} w \quad \text{so } \Omega = \{w > 0\}$$

Example:

$$k > 1 \quad \min_{\substack{r \in \mathbb{D} \\ |r| \leq \epsilon}} \lambda_k(r) \\ + \text{something}$$

Assume that we work in a family of domain satisfying a flat
cone condition



such that



$\Rightarrow$ Capacity density condition satisfied $\Rightarrow$ (E)

Assume now that we work in the family of convex sets?
(Ω_k) minimizing sequence and $\Omega_k \xrightarrow{H.C.} \Omega$
convex?

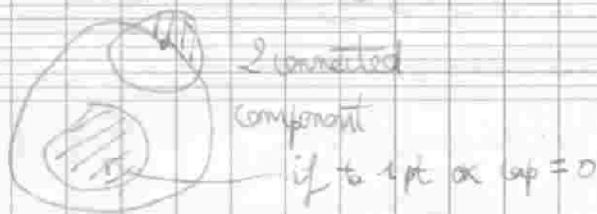
$$|\Omega_k| \leq \liminf |\Omega_k| \leq c$$

$\Rightarrow$ Capacity density condition

Theorem: (Serrin, 1992) 2-D remake
 $\uparrow$ CRADUE

In $\mathbb{R}^2$ with L^1 and $\Omega_k \subseteq D$ such that $\#^C \Omega_k \leq p$
and $\Omega_k \xrightarrow{H.C.} \Omega$ then we have $\left. \begin{array}{l} \#^C \Omega \leq p \\ \text{and} \\ \Omega_k \xrightarrow{x-w} \Omega \end{array} \right\}$

Proof: It is a consequence of the capacity density condition



III. General existence results:

Idea: $\min_{\substack{|u| \leq c \\ \Omega \subseteq \mathbb{D}}} J(\Omega)$

J depends on Ω through the Dirichlet - Laplacian

Take $(\Omega_k)_k$ a minimizing sequence

$$F_k(\omega) = \begin{cases} \int_{\Omega_k} |\nabla \omega|^2 & \text{if } \omega \in H^1_0(\Omega_k) \\ +\infty & \text{if not} \end{cases}$$

By topological arguments, there exists $F_k \xrightarrow[k \rightarrow \infty]{F(\omega)} F \leftarrow$ some functional
 $u_{k,i,j}$ minimizes $\rightarrow$ minimizes of F

Question: What is F ? How F looks like?

$$F(\omega) = \begin{cases} \int_{\mathbb{D}} |\nabla \omega|^2 + \int_{\mathbb{D}} u^2 d\mu / \text{positive measure} \\ +\infty \end{cases}$$

Construction of μ : $u_k = u_{k,i,j}$ Assume $u_k \xrightarrow{H^1_0(\mathbb{D})} u$

Let $\{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{D})$ and assume $u_{k,i,j} \xrightarrow{H^1_0(\mathbb{D})} u$ (We are going to see that u is independent of f_k)

Properties of the function u : $u_k = u_{k,i,j}$ $u_k \geq 0$ (maximum principle)
 $u_k > 0$ in Ω_k p.e.

$$-\Delta u_\varepsilon = 1 \text{ in } \mathbb{R}^n$$

$$-\Delta u_\varepsilon \leq 1 \text{ in } D?$$

The answer is yes if $\mathbb{R}^n$ is smooth. Proof: $\varphi \in C_c^\infty(\mathbb{D})$, $\varphi \geq 0$
 $\langle \Delta u_\varepsilon + 1, \varphi \rangle_{D', D} \geq 0?$

$$\begin{aligned} \int_D \varphi - \int_D \Delta u_\varepsilon \varphi &= \int_D \varphi - \int_{\mathbb{R}^n} \Delta u_\varepsilon \varphi \\ &= \int_D \varphi - \int_{\mathbb{R}^n} \varphi - \int_{\mathbb{R}^n} \frac{\partial u_\varepsilon}{\partial n} \varphi \\ &= \int_{\partial D} \varphi - \int_{\partial D} \frac{\partial u_\varepsilon}{\partial n} \varphi \geq 0 \end{aligned}$$

So, we showed that $\Delta u_\varepsilon + 1$ is a positive distribution in $H'(\mathbb{D})$
 so it is a Radon measure.

Take $\psi \in C_c^\infty(\mathbb{D})$ such that $u_\varepsilon \psi \in H_0^1(\mathbb{D}_\varepsilon)$

$$\int_{\mathbb{D}} \Delta u_\varepsilon \psi = \int_{\mathbb{D}} f(u_\varepsilon \psi)$$

$$\int_D \Delta u_\varepsilon \psi + \int_D \Delta u_\varepsilon u_\varepsilon \psi = \int_D f(u_\varepsilon \psi)$$

①

$$\int_D \Delta u \psi = \int_D f(u \psi)$$

$$\begin{aligned} \textcircled{1} &= \int_D \Delta u_\varepsilon \psi = - \int_D u_\varepsilon \Delta \psi + \langle \Delta u_\varepsilon, \psi \rangle_{H^{-1}, H^1} \\ &\quad \downarrow \quad \quad \quad \downarrow \\ &= - \int_D u \Delta \psi + \int_D \Delta u \end{aligned}$$

$$\text{So we get: } \int_D \Delta u \psi = - \int_D u \Delta \psi + \int_D \Delta u = \int_D f(u \psi)$$

We have: $\int_D \nabla u \nabla \psi - \int_D u \nabla \psi \cdot \nu + \int_D \psi u = \int_D f \psi$

$\int_D \nabla u \nabla \psi + \int_D \nabla u \cdot \nu \psi + \langle \Delta u, u \rangle_{H^1(D), H_0^1(D)} + \int_D \psi u = \int_D f \psi$

$\int_D \nabla u \nabla(\psi \varphi) + \langle \Delta u + 1, u \varphi \rangle_{H^1(D), H_0^1(D)} = \int_D f \psi \varphi$

$\int_D \nabla u \nabla(\psi \varphi) + \int_D u \varphi \frac{d\psi}{dx} = \int_D f(\psi \varphi)$

$\int_D \nabla u \nabla(\psi \varphi) + \int_D u(\psi \varphi) dx = \int_D f(\psi \varphi)$

So we get
$$\begin{cases} -\Delta u + u \cdot \mu = f \\ u \in H_0^1(D) \cap L^2(D, \mu) \end{cases} \quad \begin{array}{l} \mu \text{ positive Borel} \\ \text{measure} \\ \text{possibly taking value } +\infty \end{array}$$

μ absolutely continuous with respect to capacity

NB: μ does not depend on f
it depends only on geometry

To get the equation, as we are in $H_0^1(D) \cap L^2(D, \mu)$, you have to show that $\{f \psi, f \in C_c^\infty(D)\}$ is dense in $H_0^1(D) \cap L^2(D, \mu)$ for the norm $\|u\|_{H_0^1(D)} + \|u\|_{L^2(D, \mu)}$.

Particular cases:

$\star \quad \mu \xrightarrow{r \rightarrow 0} \nu$

$\mu = \mu_r = \nu \quad \text{where} \quad \nu(A) = \begin{cases} 0 & \text{if } A \cap \partial D = \emptyset \\ 1 & \text{if } A \cap \partial D \neq \emptyset \end{cases}$
Has if $\text{Cap}(A \cap \partial D) > 0$

*** Ciarra - Murat $\mu = \begin{cases} \text{c.d.x.} & \text{on } \Omega \\ +\infty & \text{on } \Omega^c \end{cases}$

Remarks: For μ , we define $R_\mu: L^2(\Omega) \rightarrow L^2(\Omega)$
 $f \mapsto R_\mu f = u_\mu f$

$$\text{If } \Omega \xrightarrow{\text{c.v.}} \mu \Rightarrow R_{\Omega} \xrightarrow{L^2(\Omega)} R_\mu$$

$$\Rightarrow \lambda_k(\Omega) \rightarrow \lambda_k(\mu) = \min_{\substack{S \subseteq H^1_0(\Omega, \mu) \\ \dim S = k}} \max_{u \in S} \frac{\int |\nabla u|^2}{\int u^2}$$

Exercise: Prove that $\min_{\substack{\Omega \subset \mathbb{R}^n \\ |\Omega| \leq c}} \lambda_1(\Omega)$ has a solution

Proof: (Ω_n) minimizing sequence $\Omega_n \xrightarrow{\text{c.v.}} \mu \uparrow_{\text{a.s.}} +\infty$

$$\lambda_1(\Omega_n) \rightarrow \lambda_1(\mu) \geq \lambda_1(\{w > 0\})$$

solution $\{w < 0\} \leq c$



*** Example: $\Omega \xrightarrow{\text{c.v.}} \mu \Leftrightarrow \omega_{\Omega}$ where $\Omega = \{w > 0\}$

This is equivalent to prove that $\mu(\{w > 0\}) = 0$

Indeed, take w as a test function for Ω

$$\int |\nabla w|^2 = \int w$$

$$-\Delta w - w\mu = 1$$

$$\int |\nabla w|^2 + \int w\mu = \int w$$

so we get $\boxed{\int w\mu = 0}$

Tuesday 7/6/12

BERND KAWOH

11:00 - 12:30

$$\left. \begin{array}{l} \Delta_p u_p + \lambda_p |u_p|^{p-2} u_p = 0 \quad \text{in } \Omega \\ u_p = 0 \quad \text{on } \partial\Omega \end{array} \right\}$$

$\Omega \subseteq \mathbb{R}^N$ bounded, $p > 1$

There exists a solution that minimizes $R_p(u) = \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p}$ on $W_0^{1,p}(\Omega)$

u_p is unique up to scaling with regularity $C^{1,\alpha}(\overline{\Omega})$

Cheeger constant of Ω : $\lim_{p \rightarrow 1} \lambda_p(\Omega) = h(\Omega) = \inf_{\substack{\text{domain } D \\ D \subset \Omega}} \frac{|\partial D|}{|D|}$

Why this name? Cheeger (1970)

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p$$

For $p=2$ Cheeger, general p Ambrose and Lofgren and Wei (01), Takemura (98), Matsui (2000)

So, $\liminf_{p \rightarrow 1} \lambda_p(\Omega) \geq h(\Omega)$

D_ε minimizing sequence of domain for h



$$\chi_{D_\varepsilon}^\varepsilon = \frac{1}{\varepsilon} (1 - d(x, D_\varepsilon))^+$$

$$\lambda_p(\Omega) \leq R_p(\chi_{D_\varepsilon}^\varepsilon) \leq \frac{|\partial D_\varepsilon|}{|D_\varepsilon|} \varepsilon^{1-p}$$

as $p \rightarrow 1$ then $h \rightarrow \infty$ $\limsup_{p \rightarrow 1} \lambda_p(\Omega) \leq h(\Omega)$

What happen to PDE when $p \rightarrow 1$?

In intrinsic coordinates, we have: $\Delta u = u_{\eta\eta} + (m-1)H(x)u_\eta$
 where $\eta(x) = -\frac{\nabla u}{|\nabla u|}$ (exterior normal)

$H(x) =$ mean curvature of a level set $\{u=t\}$ of u
 $u_\eta = -|\nabla u|$

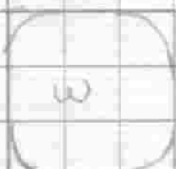
$$\Delta_p u = |\nabla u|^{p-2} \left((p-1)u_{\eta\eta} + (m-1)H(x)u_\eta \right)$$

Formally, as $p \rightarrow 1$, Euler equation becomes

$$\Delta_p u + \lambda_p |u|^{p-2} u = 0 \quad \rightarrow \quad -(m-1)H(x) + \lambda_1 = 0$$

All level sets of u should be of mean curvature $\frac{\lambda_1}{m-1}$

Example: $m=2 \rightarrow$ circular arc of length $\frac{1}{h(x)}$

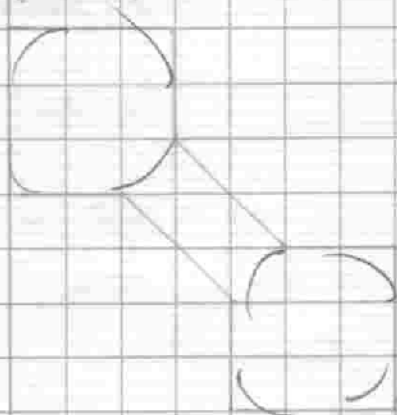


Choisy set
for a square

$p=1$ difficult to compare
for the p -Laplacian
($p \rightarrow \infty$ close to ∞ for numerical)



$p=1,15$
plateau phenomena



sets convex $\Rightarrow$ Choisy sets unique
and convex

Theorem: As $p \rightarrow 1$ $\lambda_p(\Omega) \rightarrow h(\Omega) := \text{Cheeger constant}$

$u_p(\lambda) \rightarrow u_1(\lambda)$ in $L^1(\Omega)$

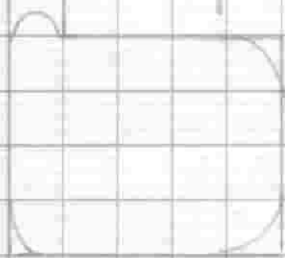
Moreover, each level set of u_1 is Cheeger set of Ω .

Definition: Here, a Cheeger set ω is a minimizing subdomain of the variational problem

$$h(\Omega) = \inf_{\omega \subset \Omega} \frac{|\partial \omega|}{|\omega|} = \frac{P(\omega, \mathbb{R}^n)}{|\omega|}$$

Remark: $R_1(\Omega) = \inf_{\omega \subset \Omega} |\partial \omega| = \int_0^\infty P(\Omega, \mathbb{R}^n, \lambda) d\lambda = \int_0^\infty |\partial \Omega_\lambda| d\lambda$
 $\int_{\partial \omega} |\nu| d\mathcal{H}^n = \int_0^\infty \text{Vol}(\Omega_\lambda) d\lambda = \int_0^\infty |\Omega_\lambda| d\lambda$

For convex $\Omega \subset \mathbb{R}^2$, Cheeger set ω is unique but for non-convex plane domain Ω , there can be non-uniqueness of ω .



Definition: $C_\Omega \subset \Omega$ is called Cheeger set of Ω if it satisfies

$$h(\Omega) = \frac{|\partial C_\Omega|}{|C_\Omega|}$$

Theorem: (Kawohl and Friedmann)

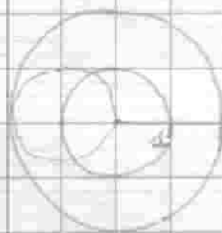
If Ω has only one Cheeger set, if $\|u_p\|_{L^1} = 1$, then $u_p \rightarrow u_1 = \chi_{C_\Omega}(\lambda)$ in $L^1(\Omega)$

Theorem: (Kakuhl and Lachand-Robert)

If $\Omega \subseteq \mathbb{R}^2$ is convex, then it has a unique Cheeger set.
 In this case, C_Ω is the unique of all balls of radius $\rho = \frac{1}{h(\Omega)}$ which are subsets of Ω .

In particular, $C_\Omega = \Omega$ if the curvature of $\partial\Omega$ does not exceed $h(\Omega)$.

In any case, the union of the union of those balls i.e. the set $\{x \in \Omega, d(x, \partial\Omega) \geq \rho\}$ has the same area as the ball of radius ρ .



If $\Omega = B_1 \subseteq \mathbb{R}^2$

Then $\begin{cases} C_\Omega = B_1 \\ h = \frac{2\pi}{\pi} = 2 \end{cases}$



What about Cheeger sets in higher dimensions?

Open problem: Explicitly calculate Cheeger sets for cube decomposition

Application in continuum mechanics (1989 SIAM Review 2 pages)

Plate failure under pressure

line of fracture

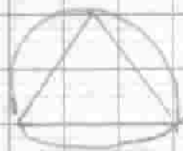
constant curvature



Convex sets with constant width

and Meissner's
mysterious bodies

Bernd Knafl (University of Cologne)



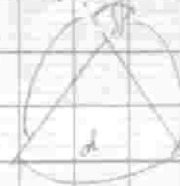
can drill
a square with it

There are many convex sets of
constant width

This support function solves $p(\theta) + p(\theta + \pi) = d$



(Construction in 2D of a set of constant
width without rotational symmetry)



Theorem: (Barin 1866) All convex plane sets of constant width d
have the same perimeter πd

NB: By the isoperimetric inequality, the disk maximizes
area among those.

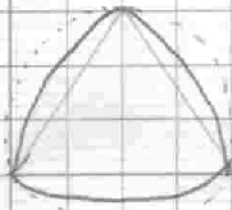
Consequently, you need more gold for irregular gold-coins than for
other of same width.

PROOF:
$$Per = \int_0^{2\pi} p(\theta) d\theta = \int_0^{\pi} \underbrace{(p(\theta) + p(\theta + \pi))}_{=d} d\theta = \pi d$$

Question: Which coin requires the least amount of gold?

Theorem: Among all plane convex sets of constant width d , the Reuleaux triangle minimizes area.

$$\min A(p) = \int_0^{2\pi} (p^1(\theta) - p^1(0)) d\theta$$



The Reuleaux triangle (red) and disk (dot) is 20% less area than the disk

Meissner (1912) body

Reuleaux of tetrahedron
no symmetry

Theorem: (Blaschke 1915) Among 3-D convex bodies of given width, the ball maximizes volume and surface area and the one that minimizes volume also minimizes surface area.

Open conjecture: (1931, Borsari, Borchel)

The convex bodies of constant width that minimize volume are exactly Meissner's bodies

Theorem: (Campi, Cordero, Franchi 2016) Among rotational rotational Reuleaux triangles are solutions

Theorem: (Guthrie-Robert, Oudet 2006)

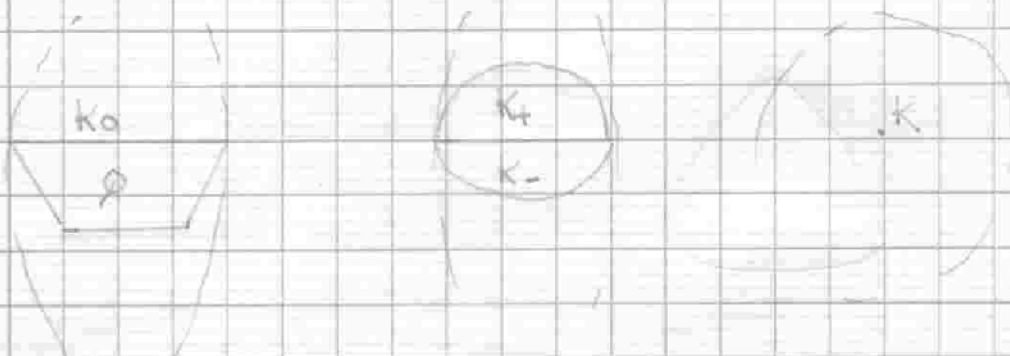
$E_+ =$ upper half-plane of $\mathbb{R}^N$

$K_0 \subseteq E_+ \cap E$ a $(n-1)$ -dimensional constant width body

and $\varnothing \subseteq \mathbb{R}^N$ satisfy $K_0 \subseteq \varnothing \subseteq E - \bigcap_{x \in K_0} B(x, d)$

Set $K_+ = E_+ \cap \bigcap_{x \in P} B(x, d)$

Then $K = K_+ \cup K_-$ is an m -dimensional constant width body with K_0 as a cross section



If $m=2$ and $\varnothing = K_0 = (a, d)$ then K is a Reuleaux triangle

If $m=3$ and $\varnothing = K_0 =$ Reuleaux triangle then K is a Meissner body

So there is a good reason to see that Meissner bodies are good candidate

Last result: (Börger, Lachand-Robert, Vudret 2007)
ARMA

It gives a necessary condition that characterizes a 3-D constant width body of (body) minimal volume

If one squashes 2 planes, then 2 tangent points it is not smooth (Meissner bodies satisfies this)

Amiroux - Guilfoyle (2009)

Das (2011)

→ www.mi.uni.koeln.de/~Kawohl

Jeudi 7/6/2 } $J(u) \leq J(v) + \lambda (|\Omega u| - m)^+ \quad \forall u \in H^1_0(D)$

NICOLAS PIERRE

15:30-17:00

} $|\Omega u| = m$

with $J(v) = \frac{1}{2} \int_D |\nabla v|^2 - \int_D f v$ and $f \in C^0(D)$

Theorem: u is locally Lipschitz continuous in D

Lemma: (i) u is continuous in D ADAMS



(ii) Δu^+ and Δu^- are measures with
 $\forall x \in D \cap D_0, |\Delta u| (B(x, r)) \leq C r^{N-1}$

Proof of (i) when $u \geq 0$:

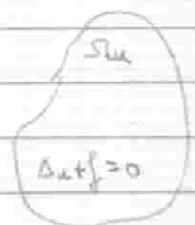
$\forall \varphi \in C_c^\infty(D) \quad v = u + t\varphi$

$$\begin{aligned} \frac{1}{2} \int_D |\nabla v|^2 - \int_D f v &\leq \frac{1}{2} \int_D |\nabla(u + t\varphi)|^2 - \int_D f(u + t\varphi) \\ &\quad + \lambda (|\Omega v| - |\Omega u|) \\ &\leq \frac{1}{2} \left(\int_D |\nabla u|^2 + 2t \int_D \nabla u \nabla \varphi + t^2 \int_D |\nabla \varphi|^2 \right) \\ &\quad + \lambda |\Omega v| - \int_D f u - t \int_D f \varphi \end{aligned}$$

We get for $t > 0$, $- \int_D \nabla u \nabla \varphi + \int_D f \varphi \leq \frac{1}{2} \int_D |\nabla u|^2 + \frac{\lambda |\Omega u|}{t}$

$\langle \Delta u + f, \varphi \rangle \leq C \|\nabla \varphi\|_{L^2(D)} |\Omega u|^{\frac{N-2}{2}} \quad \left(\begin{array}{l} g(t) = a + \frac{b}{t} \\ g'(t) = -\frac{b}{t^2} \\ g(t) \geq \inf_{t>0} g(t) \end{array} \right)$

1D 



$u=0$

$$\Delta u + f|_{\partial\Omega} \geq 0$$

If u is positive, choose $\varphi = 1$ on $B_r = B(x_0, r)$

$$\text{supp } \varphi \subseteq B(x_0, 2r)$$

$$0 \leq \varphi \leq 1$$

$$\|\Delta \varphi\|_{L^\infty(B(x_0, 2r))} \leq \frac{C}{r}$$

$$\text{We have } \int_{B_r} |\Delta \varphi|^2 \leq C r^{N-1}$$

$$\|\Delta \varphi\|^{1/2} \leq C r^{N/2}$$

$$\text{So we get } \langle \Delta u + f, \varphi \rangle \leq C r^{N-1}$$

$$\langle \Delta u + f|_{\partial\Omega}, \varphi \rangle \leq C r^{N-1} + \int_{\partial\Omega} (f|_{\partial\Omega} - f) \varphi$$

$u \geq 0$

$\Rightarrow \Delta u + f|_{\partial\Omega}$ positive measure

$$|\Delta u + f|_{\partial\Omega}|(B_r) \leq C r^{N-1} + \int_{B_r} (f|_{\partial\Omega} - f) \varphi \leq C \|f\|_{L^\infty(\mathbb{R}^N)} r^N \quad (\text{PASCAL})$$

For general u , $\langle \Delta u^+ - \Delta u^- + f, \varphi \rangle \leq C r^{N-1} + \text{use the positivity lemma}$

Lemma: $u \in C^2(B(x_0, r_0)) \quad \forall r \leq r_0$

$$\int_{\partial B(x_0, r)} (u - u|_{\partial B(x_0, r)}) = C r \int_0^r \int_{\partial B(x_0, s)} \Delta u \, ds$$

Proof: $\frac{d}{dr} \int_{\partial B(x_0, r)} = \dots$ change of variables on $\partial B(x_0, r)$
 $\Delta u + \text{integration by parts} + \frac{2}{r} \int_{\partial B(x_0, r)} u \, ds$

Remark: It remains true when $\int_0^r \int_{\partial B(x_0, s)} |\Delta u| \, ds < +\infty$
 This is the case for our function

$$-\Delta u \leq g \in L^0(\mathbb{R}^N)$$

NB: Assume that $|\Delta u|(B(x_0, r)) \leq C r^{N-1}$

$\int_{\partial B(x_0, r)} (u - u(x_0)) \leq C r \int_0^r s^{N-1} r^{N-1} \leq C r^N$

$u = |u| \quad u(x_0) = 0$



$\int_{\partial B(x_0, r)} |u| - 0 \leq C r \quad (\text{A REFINER})$

Lemma: $F \in L^\infty(B(x_0, r_0))$. Then $\forall r \leq r_0$

① If $-\Delta u = F$ on $B(x_0, r)$, then $\|\nabla u\|_{L^\infty(B(x_0, \frac{r}{2}))} \leq C \left(1 + \frac{1}{r} \|u\|_{L^\infty(B(x_0, r))}\right)$

② If $\begin{cases} -\Delta u \leq F \\ u \geq 0 \end{cases} \Rightarrow \|u\|_{L^\infty(B(x_0, \frac{r}{2}))} \leq C \left(r + \int_{\partial B(x_0, r)} u\right)$

Proof lemma: ① $\begin{cases} -\Delta v = 0 & \text{on } B(x_0, r) \\ v = u & \text{on } \partial B(x_0, r) \end{cases}$

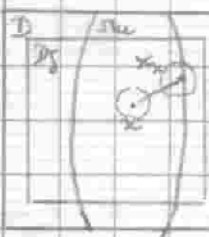
 $F \equiv 0$
use the Poincaré inequality
harmonic on a ball

② $\begin{cases} -\Delta w = F & \text{on } B(x_0, r) \\ w = 0 & \text{on } \partial B(x_0, r) \end{cases}$

③ $\|u\|_{L^\infty} \leq \|v\|_{L^\infty} + \|w\|_{L^\infty}$ where $\begin{cases} -\Delta z = F & \text{on } B(x_0, r) \\ z = u & \text{on } \partial B(x_0, r) \end{cases}$

back to ①

Proof of Lipschitz continuity of u :



$\nabla u(x) = 0$ for a.e. $x \in D \setminus \partial D$

As u is continuous, ∂D is open

$x \in \partial D \cap D_\delta$
 $x_x \in \partial D$ be
such that $|x - x_x| = d(x, \partial D) = d_x$

$$u(x_0) = 0 \quad B(x_0, dx) \subseteq \Omega_u$$

$$\textcircled{1} \quad dx \geq \frac{\varepsilon}{16} \quad B(x, \frac{\varepsilon}{16}) \subseteq \Omega_u \quad -\Delta u = f$$

$$\text{lemma} \Rightarrow \|\nabla u(x)\|_{\infty} \leq C\varepsilon \quad \text{Indeed}$$

$$\text{lemma says } |\nabla u(x)| \leq C \left(1 + \frac{1}{dx} \|u\|_{L^\infty(B(x, dx))} \right) \quad \textcircled{2}$$

As $B(x_0, 2dx)$ is
no more included
in Ω_u we only
know $-\Delta u \leq |f|$
on $B(x_0, 2dx)$

$$\begin{aligned} \|u\|_{L^\infty(B(x_0, dx))} &\leq \|u\|_{L^\infty(B(x_0, 2dx))} \\ &\leq C(dx + \int_{B(x_0, 2dx)} |f|) \\ &\leq C dx \end{aligned}$$

we proved $|\Delta u|(B(x_0, r)) \leq C r^{N-1}$
then $u(x_0) = 0 \Rightarrow \int_{B(x_0, r)} |f| \leq C r^N$

Some steps to obtain the regularity of the boundary $\partial \Omega_u$ itself

$$u \equiv 0$$

$$\Delta u + f = 0$$

$$\frac{\partial u}{\partial n}$$

$$|\nabla u| = \lambda \min$$

(Euler-Lagrange equation)
if regularity

$$f \geq 0$$

here

$\partial^+ \Omega_u \cap D$ is a regular hypersurface

$$\int_{\partial^+ \Omega_u} (|f|) \partial^+ \Omega_u = 0$$

$N=2$, $\partial \Omega_u$ is regular

GOAL $\Rightarrow$ it is to prove that $\Delta u + f|_{\partial \Omega_u} = \lambda \frac{\partial^2 u}{\partial n^2} |_{\partial \Omega_u}$
If this is true, then it is ok
(Caffarelli works)

The main lemma is the following:

$$C_1, C_2 \geq 0 \quad C_1 r^{N-1} \leq \int_{\partial \Omega_u} |\Delta u + f| \chi_{\partial \Omega_u} (B(x_0, r)) \leq C_2 r^{N-1}$$

This lemma implies $\forall R \subset \mathbb{R}^n$ compact, $\mathcal{H}^{n-1}(R) \leq C \mu_n(R)$

Proof of this implication: $\eta > 0$ $K \subseteq \bigcup_{\substack{z \in K \\ r \in (0, \eta)}} B(z, r)$

There exists Vitali subcovering: countable $B(z_i, r_i)$ $0 < r_i < \eta$ disjoint and $K \subseteq \bigcup_i B(z_i, 5r_i)$

$$\sum_i r_i^{n-1} \leq \sum_i C \mu_n(B(z_i, r_i)) \leq C \mu_n(\{x \in \mathbb{R}^n, d(x, K) \leq \eta\})$$

$$\mathcal{H}_\eta^{n-1}(K) \leq \frac{1}{5^{n-1}} \sum_i (5r_i)^{n-1}$$

Then, as $\eta \rightarrow 0^+$, we get $\mathcal{H}^{n-1}(K) \leq C \mu_n(K)$

Thus, μ_n is absolutely continuous with respect to $\mathcal{H}^{n-1}$

By Radon-Nikodym theorem, there exists a density

$$q_n \in (L^1(\mathbb{R}^n, d\mu_n)) \text{ such that } \mathcal{H}^{n-1}|_{\mathbb{R}^n} = q_n d\mu_n$$

Now, the goal is to show that the density q_n is constant.

The lemma gives $C_1 \leq q_n \leq C_2 \Rightarrow \mu_n = \frac{1}{q_n} \mathcal{H}^{n-1}|_{\mathbb{R}^n}$ one step is to replace with μ_n

The last step is to show $\frac{1}{q_n} = 1$

PROOF OF LAST STEP

$$\forall \varphi, \int \varphi(\Delta u + f) = \int_{\mathbb{R}^n} \varphi \frac{1}{q_n} d\mathcal{H}^{n-1}$$

Next step (blow-up argument)

$$\varphi(r) = \frac{1}{r^m} \varphi(x_0 + r\omega)$$

$$u_n(r) = \frac{1}{r^m} u(x_0 + r\omega)$$



Liouville theorem $\Rightarrow u$ is linear
 $\Delta u = f = -\Delta u$

Application: $\lambda_1(\hat{\Omega}) = \min \{ \lambda_1(\Omega), \Omega \subseteq D, |\Omega| = m \}$

Theorem: Assume that D is connected bounded, then

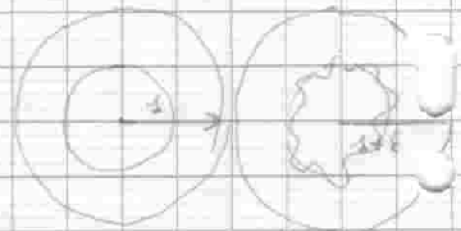
u_1 is locally Lipschitz $u_1 \geq 0$

$\hat{\Omega}$ is of finite perimeter, $\chi_{\hat{\Omega}}$ is regular and $H^{N-1}(\partial \hat{\Omega}) < \infty$
in 2-D, it is fully regular ($\chi_{\hat{\Omega}}$)

NB: If $D = \bigcirc_{D_1} \bigcirc_{D_2}$ $m = \pi + \epsilon$

For the optimal λ_1 , $D_1 \subseteq \hat{\Omega}$

For λ_2 , $m = \pi + \pi(1+\epsilon)^2$ $\lambda_2(\hat{\Omega}) \geq \max(\lambda_1(D_1), \lambda_1(D_2)) \geq \lambda_1(D_2)$
 $\hat{\Omega} = D_1 \cup D_2$

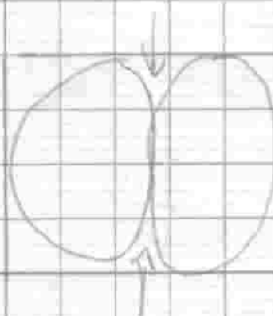


$D_1 = D_1$

$\lambda_1(D_1) > \lambda_1(D_2)$

$D_2 = D_2 + \epsilon$

bounded domain which is optimal
have eigenfunction very bad
although for λ_1 it was Lipschitz



we can prove that $\hat{\Omega}$
connected, then u_1 is
Lipschitz

2 connected components

Friday 8/6/12 11/ A few words about optimal shapes for $E(\Omega) = P(\Omega) + J(u_\Omega)$

MICHAEL PERLIS

3:00-10:30

$$J(u_\Omega) = \frac{1}{2} \int_\Omega |\nabla u_\Omega|^2 - \int_\Omega f u_\Omega \quad f \in L^q(\Omega)$$

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

$$E(\hat{\Omega}) = \min \{ E(\Omega) : \Omega \subseteq D, |\Omega| = m \} \quad (P)$$

① This is equivalent to the perimeter version (again) (Assume D is star-shaped)

If $\hat{\Omega}$ satisfies (P), then for λ large enough we have

$$E(\hat{\Omega}) \leq E(\Omega) + \lambda |\Omega - m| \quad \forall \Omega \subseteq D$$

PROOF: If $E = P$, then the optimal shape is a quasi-minimizer (theory of minimal surfaces to get regularity)
But here, $E \neq P$

② We can prove that χ_Ω is $\frac{1}{2}$ -Hölder continuous

$$|\chi_\Omega| \chi_{\Omega'}(B(x_0, r)) \leq C r^{N-1-\frac{1}{2}} \quad \text{in positive case if } \chi \text{ changes sign, use continuity lemma}$$

(C^2 would be enough)

③ $f \geq 0, f \in C^1(\bar{D})$, then χ_Ω is Lipschitz continuous

PROOF: More difficult, perturbation by parabola

Then, we are able to prove that $\hat{\Omega}$ is a quasi-minimizer
(of (1.1)) and get the $C^{1,\alpha}$ -regularity of its reduced
boundary.



$$|\Omega \Delta \hat{\Omega}| \leq C B(x_0, r)$$

$$P(\hat{\Omega}) + \int_{\hat{\Omega}} u \leq P(\Omega) + \int_{\Omega} u + \chi | \Omega - \hat{\Omega} |$$

$$\leq \int_{\Omega} u$$

Let be $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$\begin{cases} \varphi \equiv 0 \text{ on } B(x_0, r) \\ \varphi \equiv 1 \text{ outside } B(x_0, 2r) \\ 0 \leq \varphi \leq 1 \\ \|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{r} \end{cases}$$

$$P(\hat{\Omega}) - P(\Omega) \leq \frac{1}{2} \int_{\Omega} |\nabla(u \varphi)|^2 - |\nabla u|^2 + \int_{\Omega} f(u - u \varphi) + C r^N$$

$$\leq \frac{1}{2} \int_{\Omega} u^2 |\nabla \varphi|^2 + \varphi^2 |\nabla u|^2 + 2 \int_{B(x_0, 2r)} u \varphi |\nabla \varphi| |\nabla u| - |\nabla u|^2$$

$$+ \int_{\Omega} f(u(1 - \varphi)) + C r^N$$

$$\leq \frac{1}{2} \int_{\Omega} (\varphi^2 - 1) |\nabla u|^2 + \int_{B(x_0, 2r)} u^2 |\nabla \varphi|^2 + \int_{B(x_0, 2r)} \varphi^2 |\nabla u|^2$$

$$+ \int_{\Omega} f(u(1 - \varphi)) + C r^N$$

$$\textcircled{1} \leq 0$$

$$\textcircled{2} \leq \frac{C}{r^2} r^N r^2 \xrightarrow{\text{not exact}} |u(x) - u(y)| \leq \|\nabla u\|_{L^2(\Omega)} |x - y| \quad \forall x, y \in B(x_0, 2r)$$

$$\|\nabla u\|_{L^2(\Omega)} \leq C r$$

$$\textcircled{3} \leq C \|\nabla u\|_{L^2(\Omega)}^N \leq C r^{N-1}$$

$P(h) = P(r) \leq C r^N$ we needed $C r^{N-1+\kappa}$ $\kappa \in (0,1)$ or iii

(Si on fait du $\frac{1}{2}$ -Hölder continuous dans le même raisonnement cela donne $C r^{N-1}$. On est sûr, il faut $\frac{1}{2}$ -Hölder continuous)

IV/ Optimization with convex constraints: in $\mathbb{R}^D$ an analytic point of view

$\min \{ E(\Omega), \Omega \subseteq \mathbb{R}^2, \Omega \text{ convex} + \text{possible constraints} \}$



$$\omega(\theta) = \frac{1}{\rho(\theta)}$$

$$\text{convexity} \Leftrightarrow \omega(\theta) = \frac{\omega + \omega''}{(1 + (\omega'/\omega)^2)^{3/2}} \geq 0$$

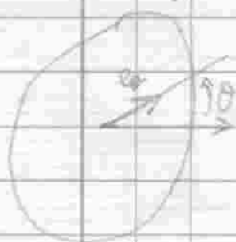
convexity $\Leftrightarrow \omega + \omega'' \geq 0$ it is a linear constraint

$\Omega \cap [\theta_1, \theta_2] = \text{segment} \Leftrightarrow \omega + \omega' = 0$ on (θ_1, θ_2)

Ω polygon $\Leftrightarrow \omega + \omega'' = \text{finite sum of Dirac masses}$

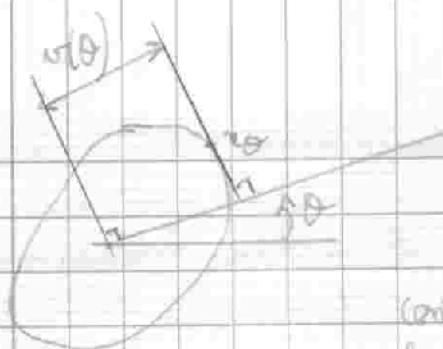
Our shape problem is transformed into an analytical problem

$\min \{ E(\Omega) = f(\omega), \omega \in W^{1,2}(\mathbb{S}^1), \omega + \omega'' \geq 0 \}$
+ possible constraints



$$\omega(\theta) = \sup \{ \langle n, e_\theta \rangle, n \in \mathcal{N}, e_\theta = (\cos \theta, \sin \theta) \}$$

This is called the support function



$$(v+v'')(a) = \frac{1}{\text{curvature}(v)}$$

convexity $\Leftrightarrow v+v'' \geq 0$
 polygons $\Leftrightarrow v+v'' = \text{finite sum of Dirac masses}$

$$j(v) = \int_{\mathbb{R}^1} G(v(a), v'(a)) da \quad G: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ regular (or local)} \\ (v, v') \mapsto G(v, v')$$

$$\left\{ \begin{array}{l} j(u) = \min \{ j(v), v \in W^{1,\infty}(\mathbb{R}^1), v+v'' \geq 0, 0 \leq a \leq u \leq b \} \\ v+v'' \geq 0 \\ 0 \leq a \leq u \leq b \end{array} \right.$$

Theorem 1: If G is strictly convex with respect to v' , then the optimal function is in $W^{2,\infty}(\mathbb{R}^1 \cap [a, u] \cup [u, b])$

Theorem 2: If G is strictly concave with respect to v' , then $\text{supp } u \cap K$ is a polygon $\forall K \text{ compact} \subseteq [a, u] \cup [u, b]$

A main tool: the optimality condition including Lagrange multiplier for convexity constraint

$$a) \exists \eta \in W^{1,\infty}(\mathbb{R}^1), \forall v \in W^{1,\infty}(\mathbb{R}^1)$$

$$\text{supp}(u+v'') := \text{Su}$$

$$j'(u)v = (\eta + \eta'', v) = \int_{\mathbb{R}^1} \eta v - \eta' v' \quad \eta \geq 0 \\ \text{supp } \eta = 0$$

PROOF: Coming from abstract results (it is right away from) $u=a / u=b$

(b) $\forall u \in W^{1,2}(\mathbb{R}^1)$ such that $(\eta + \eta'') \cdot v = 0$, $\exists \lambda \in \mathbb{R}$, $v + v'' \geq \lambda(u + u'')$ then $j'(u) \cdot (v, v') \geq 0$

Proof of theorem 1:

$$\begin{aligned} j'(u) \cdot v &= \frac{d}{dt} \bigg|_{t=0} \int_{\mathbb{R}^1} G(u+tv, u'+tv') dt \\ &= \int_{\mathbb{R}^1} G_u(u, u') v + G_{u'}(u, u') v' \\ &= \int_{\mathbb{R}^1} v [G_u(u, u') - (G_{u'}(u, u'))'] \end{aligned}$$

We have $j'(u) \cdot v = \langle \eta + \eta'', v \rangle$ so we get

$$G_u(u, u') - G_{u''}(u, u') u' - G_{u'}(u, u') u'' = \eta + \eta''$$

$$\begin{aligned} \eta &\in W^{1,2}(\mathbb{R}^1) \\ \eta &\geq 0 \end{aligned}$$

$$S_u := \text{supp}(u + u'') \subseteq [h=0]$$

$$\begin{aligned} 0 \leq u + u'' &= \mu_a + \mu_s & \mu_a &\ll d\theta & \mu_s \perp d\theta \\ \eta + \eta'' &= \nu_a + \nu_s & \nu_a &\ll d\theta & \nu_s \perp d\theta \end{aligned}$$

Lemma: $\eta'' \geq 0$ on $[h=0]$. This gives $\nu_s \geq 0$ on S_u .

PROOF: $\begin{array}{c} \nearrow p_n(s) \\ \searrow \eta_n \end{array} \quad \begin{array}{c} \eta \in C_c^\infty(\mathbb{R}^1) \\ \eta \geq 0 \end{array} \quad \begin{array}{c} \int \eta p_n(s) \eta'' = - \int \eta' p_n(s) \eta' - \underbrace{\int \eta p_n''(s) (\eta')^2}_{\geq 0} \\ \downarrow \\ \int_{\eta=0} \eta \eta'' \geq - \int_{\eta=0} \eta' \eta' = 0 \end{array}$

(2) We have then $G_u - G_{u''} u' - G_{u'}(\mu_a + \mu_s) = \eta + \nu_a + \nu_s$

If G convex with respect to u' , then $G_{u'} \leq 0$

$\Rightarrow$ opposite sign $\mu_s \equiv 0$ and $\nu_s \equiv 0$.

$$(ii) \quad 0 \leq \eta'' \quad u'' + u \geq 0 \iff -u'' \leq u$$

$$0 \leq \eta'' \leq \underbrace{G_u - G_{qu} + G_{qu} - \eta}_{\in L^\infty}$$

$$\Rightarrow G_{\eta\eta} u'' \in L^\infty \xRightarrow{\text{strictly convex}} \underline{u'' \in L^\infty}$$

Remarks: It is a local result (separate convex and concave part of η)
We can "localize".

.. It can be extended to functional of the form

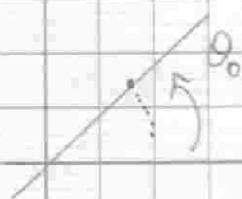
$$E(u) = F(|u|, J(u), \lambda(u)) \pm \int_{\Omega} P(u)$$

$$P(u) = \int_{\mathbb{R}^2} \frac{|u|^2 + u^2}{|x|^2} dx$$

it is a convex function

Proof of theorem (2): The idea comes from Lichnerowicz-Robert
while studying Newton's problem towards
talked about.

We want to prove that $u \pm u'' = \text{finite sum of Dirac masses}$
By contradiction, assume $\exists \theta_0, a < u(\theta_0) < b$ and $\theta_n \in \text{supp } u$
 $\theta_n \rightarrow \theta_0$



is perturbation with compact support as small as
you want

For those ϵ_n , we have $0 \leq j''(u)(v_n, v_n)$

$$0 \leq \int_D G_{nn} w_n^2 + 2 \int_D G_{np} w_n w_p' + G_{pp} (w_p')^2$$

$$\leq \epsilon \int_D w_n^2 + C \int_D w_p^2 \quad \text{strictly concave } (\leq -\alpha)$$

$$\int_D w_p^2 \geq C_n \int_D w_n^2 \quad \left(\begin{matrix} C_n \rightarrow +\infty \\ n \rightarrow +\infty \end{matrix} \right)$$

and $\text{supp} \rightarrow 0$

$$0 \leq \int_D G_{nn} w_n^2 + C \int_D w_n^2 + \epsilon \int_D (w_p')^2 - \alpha \int_D (w_p')^2 < 0$$

Contradiction
No accumulation points

Laframboise - Robert - Pelletier

They also did it for the problem $\min \int_D \frac{dx}{1 + |x|^{p-2}}$, $x: D \rightarrow [0,1]$

convex bodies $\Leftrightarrow$ x concave



$$\int_D \frac{dx}{1 + |x|^{p-2}} = \int_D \pi (h_1(x) - x^{p/2} h_2(x)) dx \quad h_1(x) > 0$$

N convex but x convex hull variable

Fix N and write (P) then compute the Newton's energy on convex set (the convex hull)

Friday 08/06/12

MORNING

11:00 - 12:30



$$\left\{ \begin{array}{l} -\Delta u_{\text{self}} = f \text{ in } \Omega \\ u_{\text{self}} \in H_0^1(\Omega) \end{array} \right.$$

We only have to look at $f \equiv 1$

$$u_{\text{self}} = u_{\text{self},1}$$

$$u_{\text{self}} \xrightarrow[L^2(\Omega)]{H_0^1(\Omega)} u_{\text{self}} \quad \left\{ \begin{array}{l} -\Delta u + \mu u = f \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{array} \right.$$

$$u_{\text{self}} \xrightarrow[L^2(\Omega)]{H_0^1(\Omega)} u$$

$$\mu = \frac{\Delta u + 1}{u}$$



If $\mu \equiv 0$ on $\{\omega > 0\}$, then $\mu = \infty_{\Omega}$ where $\Omega = \{\omega > 0\}$

$$\Rightarrow \mu \equiv \infty$$

Remark: $\mu \equiv \infty \iff u_{\text{self}} \xrightarrow{H_0^1(\Omega)} u$

Indeed, strong $CV \iff \int |\nabla u_{\text{self}}|^2 \rightarrow \int |\nabla u|^2$

$$\begin{array}{ll} -\Delta u_{\text{self}} = 1 \text{ in } \Omega & \times u \\ -\Delta u + \mu u = 1 \text{ in } \Omega & \times u_{\text{self}} \end{array} \quad \begin{array}{l} \int |\nabla u_{\text{self}}|^2 = \int u_{\text{self}} \\ \int |\nabla u|^2 + \int \omega^2 d\mu = \int u \end{array}$$

$$\epsilon \rightarrow 0^+ \quad \int |\nabla u|^2 = \int u = \int |\nabla u|^2 + \int \omega^2 d\mu \Rightarrow \int \omega^2 d\mu = 0$$

OK

Theorem: (Buttaro - Dal Maso 1993)

$$\mathcal{H}_c = \{ \Omega \subseteq \mathbb{R}^n, \Omega \text{ quasi-open}, |\Omega| \leq c \}$$

Assume that $F: \mathcal{H}_c \rightarrow \mathbb{R}$ such that F is decreasing with respect to inclusion and F is lower semi-continuous for δ -cv. Then,

$\min_{\Omega \in \mathcal{H}_c} F(\Omega)$ has at least one solution

This is the only result true for a large class of shape.

Remarks:

* A strong hypothesis $\rightarrow F$ decreasing

$$\Omega_1 \subset \Omega_2 \Rightarrow F(\Omega_1) \geq F(\Omega_2)$$

Examples: $F(\Omega) = \chi(\Omega)$

$$F(\Omega) = E(\Omega) = \min \left\{ \int_{\Omega} |\nabla u|^2 - f u, u \in H_0^1(\Omega) \right\} \quad \left(\begin{array}{l} \text{min of functional} \\ \text{in } H_0^1 \text{ on } \Omega \end{array} \right)$$

$$F(\Omega) = \chi(\Omega)$$

$$F(\Omega) = 5\chi(\Omega) + 10\chi(\Omega)$$

But for $F(\Omega) = \frac{\chi(\Omega)}{\chi(\Omega)}$ NOT GOOD

** The lsc χ -CV $\rightarrow$ weak hypothesis: every natural functional has

Proof of the theorem:

PARTICULAR CASE: $F(\chi(\Omega), \dots, \chi(\Omega))$

$F: \mathbb{R}^k \rightarrow \mathbb{R}$

F lsc for χ -CV
 F increasing

$(\Omega_k)_k$ minimizing sequence $\Omega_k \xrightarrow{\chi} \mu$
 $\chi(\Omega_k) \rightarrow \chi(\mu)$

$$F(\chi(\Omega_k), \dots, \chi(\Omega_k)) \leq F(\chi(\mu), \dots, \chi(\mu)) \leq \liminf_k F(\chi(\Omega_k), \dots, \chi(\Omega_k))$$

increasing lsc

Technical lemma: Assume $\Omega_k \xrightarrow{\gamma\text{-}CV} \Omega$, then $\exists (\Omega_k)$
and $\Omega_k \subseteq \hat{\Omega}_k$ such that
 $\hat{\Omega}_k \xrightarrow{\gamma\text{-}CV} \Omega$

Definition: $\Omega_k \xrightarrow{\text{weak } \gamma\text{-}CV} \Omega$ if $u_k \xrightarrow{H^1_0(D)} u$ and $\Omega := \{u > 0\}$

PROOF OF THE THEOREM IN GENERAL: Take a minimizing sequence $(\Omega_k)_k$

Up to subsequence $\Omega_k \xrightarrow{\gamma\text{-}CV} \Omega$
or $\Omega_k \subseteq \hat{\Omega}_k \xrightarrow{\gamma\text{-}CV} \Omega$

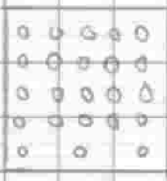
Then $F(\Omega) \leq \liminf_{k \rightarrow \infty} F(\hat{\Omega}_k) \leq \liminf_{k \rightarrow \infty} F(\Omega_k)$
decreasing

Remarks: What happens if F is not decreasing?
In general, no $\star$

Example: $\Omega \subset \mathbb{D}$ $c > 0$ is solution of $\begin{cases} -\Delta u + cu = f \text{ in } \mathbb{D} \\ u \in H^1_0(\mathbb{D}) \end{cases}$

Ciriacaru-Murat problem in $\mathbb{D}$ $\min_{\Omega \subset \mathbb{D}} \int_{\mathbb{D}} (u_{\Omega, f} - u_c)^2 dx$

From Ciriacaru-Murat, we know that infimum is 0 ($f \in V$)

Ω_k  We asked $\Omega_{k,1} \xrightarrow{H^1_0(D)} u_c$
 $\xrightarrow{L^2(D)}$
But clearly, there is no Ω such that $u_{\Omega, f} = u_c$

(Indeed, $\Omega = \mathbb{D}$ contradiction)

(NB: A_c is closed to $\gamma\text{-}CV$, this is a consequence of compact embedding of $H^1_0(\mathbb{D}) \hookrightarrow L^2(\mathbb{D})$)

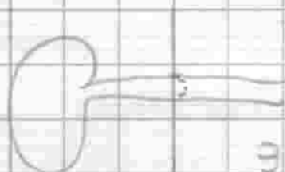
The $\Omega = \mathbb{R}^N$ case: If we remove the box, then

$$H_0^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \text{ not compact}$$

Theorem: (Mountain Pass - Ambrosetti) If $F: \mathbb{R}^N \rightarrow \mathbb{R}$ is lower semi-continuous and increasing in each variable, then there exists a bounded minimizer for the problem:

$$\min \{ F(u_1, \dots, u_k), \Omega \subseteq \mathbb{R}^N \text{ quasi-open}, |\Omega| = m \}$$

Idea of the proof:



The main ingredient is the following

$$\begin{aligned} & \exists \Omega(c, l, k, N), \forall \Omega \text{ such that } \left\{ \begin{array}{l} |\Omega| = c \\ \lambda_i(\Omega) \leq l \end{array} \right. \\ & \exists \hat{\Omega} \text{ such that } \left\{ \begin{array}{l} |\hat{\Omega}| = c \\ \dim \hat{\Omega} = N \end{array} \right. \\ & \text{and } \forall j=1, \dots, k, \lambda_j(\hat{\Omega}) \leq \lambda_j(\Omega) \end{aligned}$$

$\begin{matrix} \text{bound} & \text{bound} & \text{dimension of} \\ \text{of } c & \text{of } l & \text{of } \Omega \end{matrix}$

NB: It is independent of the functional

Different approach: concentration compactness principle of P.L. Lions

$$H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \text{ not compact (But) if } (u_n) \in H^1(\mathbb{R}^N)$$

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^N} |u_n|^2 = 1 \\ \int_{\mathbb{R}^N} |\nabla u_n|^2 \leq M \end{array} \right.$$

We have 3 possible behaviors:

* Compactness sequence: $\exists \lim_{k \rightarrow \infty} y_k \in \mathbb{R}^N$ $\lim_{k \rightarrow \infty} (y_k) \xrightarrow{L^2(\mathbb{R}^N)} u$

* vanishing: $\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} u_k^2 = 0 \quad \forall R > 0$

* dichotomy: $\exists$ double concentration of mass



Theorem: Let (α_k) be a sequence of quasi-open sets of $\mathbb{R}^N$ such that $|\alpha_k| \leq C$. There exists a subsequence such that

* compactness $\rightarrow \exists (y_k) \in \mathbb{R}^N$, $R_{y_k + \alpha_k} \xrightarrow{L^2(\mathbb{R}^N)} R$
 NB: The vanishing effect hidden here when $R_k \rightarrow 0$ (as $k \rightarrow \infty$) constructed in the same way than in Brezis-Pellé-Marcus

* dichotomy $\rightarrow \exists \alpha_k^1, \alpha_k^2 \subseteq \alpha_k$, $d(\alpha_k^1, \alpha_k^2) \rightarrow +\infty$
 $\liminf |\alpha_k^i| > 0 \quad i=1,2$ and $\|R_{\alpha_k^1} - R_{\alpha_k^2}\|_{L^2(\mathbb{R}^N)} \rightarrow 0$

Example: $\min_{\substack{|\alpha| \leq C \\ \alpha \subseteq \mathbb{R}^N}} I_2(\alpha)$ (α_k) minimizing sequence
 $R_{\alpha_k} \rightarrow R_{\alpha} \Rightarrow \{\alpha > 0\}$ is a solution!
 If dichotomy occurs

regularity Definition: A quasi-open set $\Omega \subseteq \mathbb{R}^N$ of finite measure is a shape subdomain for the energy problem if,

$$\exists \lambda > 0, \quad E(\Omega) + \lambda |\Omega| \leq E(A) + \lambda |A|, \quad \forall A \subseteq \Omega$$

(minimally bad)

$$E(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{1}{2} \int_{\Omega} |\nabla u|^2 = \frac{1}{2} \int_{\Omega} f u$$

$f=1$

Theorem: If Ω is a shape subdomain then

Ω is bounded by a constant depending on $(|\Omega|, \lambda, N)$

Ω has finite perimeter $|\partial \Omega| \leq \text{constant}(|\Omega|, \lambda, N)$

(+ inner density inside, ...)

Remarks: $\forall \Omega \subseteq \mathbb{R}^N$ of finite measure, $\exists C(|\Omega|)$ such that

$$\forall A \subseteq \Omega, \quad \frac{1}{\lambda_1(\Omega)} - \frac{1}{\lambda_1(A)} \leq C(|\Omega|) |E(\Omega) - E(A)|$$

Let's forget locality $\lambda_1(A) - \lambda_1(\Omega) \leq C(E(A) - E(\Omega))$

↑ depends of bad contents

(consequence $\rightarrow$ if Ω is a minimizer for λ_1 , then Ω is a shape subdomain for the energy!)

$A \subseteq \Omega$ proof: By minimality $\lambda_1(\Omega) + |A| \leq \lambda_1(\Omega) + |\Omega| \leq \lambda_1(A) + |A|$

$$|\Omega| - |A| \leq \lambda_1(A) - \lambda_1(\Omega) \leq C^*(E(A) - E(\Omega))$$

$$|\Omega| + C^* E(\Omega) \leq C^* E(A) + |A|$$

Theorem: Let $F: \mathbb{R}^k \rightarrow \mathbb{R}$ be increasing and locally Lipschitz.
Then for every minimizer of $F(\lambda_1(x), \dots, \lambda_k(x)) + |x|$
is a shape substitution

very important, stays
against noise, can't
remove it

Lipschitz

$\frac{1}{c} \|x\| \leq \|F(x) - F(y)\| \leq c \|x - y\|$ Let $F: \mathbb{R}^k \rightarrow \mathbb{R}$ be increasing and locally Lip-Lipschitz
then every

$\min_{|x|=c} F(\lambda_1(x), \dots, \lambda_k(x))$ is a shape substitution

Conclusion: no results for Neumann

partial for Robin